



Modeling Credit Risk: simulation of a reduced-form model

Por

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BIOGRAPHY NOTE

I have a degree and an associate one year post-graduation in apply economics at the university Paris-Dauphine (France). One year later I obtained a post-graduation in health economics at the same university. Then, I decided to follow a more quantitative domain in economics and I registered to this post-graduation in quantitative methods in economics where I´m presenting now my dissertation. I´m currently enrolled in the first year of a Phd program in economics at the University of Economics of Porto.

I decided to choose a dissertation in Credit Risk because of the European sovereign bond crisis. I wanted to understand how this Credit Risk is modeled and why it is so difficult to assess it. My desire to do a Monte-Carlo simulation drove me to produce this dissertation.

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ABSTRACT (EN)

This dissertation reviews how default probability and a valuation of a zero coupon bond are modeled in Credit Risk. After a description of different types of Credit Risk models, we argue, based on Giesecke (2004), that each model fundamentally depends on the information perceived. And this is precisely the availability of information that will characterize the event of default and will condition the valuation of a derivative.

A reduced-form model, based on Duffie and Singleton (2003), is used to give an example of a simulation of a default probability and a simulation of a zero coupon bond before default. These two simulations are made following a Monte-Carlo methodology using stochastic processes for the parameterization. The parameters are chosen from the literature (Duffie (1999), Brigo and Afonsi (2005) and Andersen and Lung (1997)).

We show that the parameters of a stochastic process have to be well specified and the volatility must be controlled. This derives obviously from the randomness of this kind of process. Reduced-form models appear to be really sensible and close to the selected sample. The simulation of a valuation of a zero coupon bond gives a sloping downward price when time is increasing. This implies that interest rate is increasing throughout time according to Mishkin (2010). Consequently one can observe that the reduced-form model incorporate a premium in order to bear more risk as time goes up. Those Monte-Carlo simulations are precise; with confidence intervals that are very close to the mean path. We conclude that reduced-form models may excessively emphasize on the fact that information is only stochastic. Maybe, as Giesecke (2004) stipulates, an incomplete information model can be a better solution, at least in theory, to improve the reduced-form models which are too close to the sample.

ABSTRACT (PT)

Esta dissertação examina como é que a probabilidade de insolvência e a estimação de uma obrigação de cupom zero são modeladas no âmbito de um risco de crédito. Após uma descrição de diferentes tipos de modelos de risco de crédito, apresentamos, com base no artigo de Giesecke (2004), que cada modelo depende fundamentalmente da informação captada. É precisamente a disponibilidade desta informação que caracteriza o evento de insolvência e que condiciona a estimação de uma obrigação.

Um modelo de forma reduzida, desenvolvido pelo Duffie e Singleton (2003), é usado para dar um exemplo de simulação da probabilidade de insolvência e um exemplo de simulação de uma obrigação de cupom zero antes do acontecimento da insolvência. Estas duas simulações são elaboradas segundo uma metodologia de Monte-Carlo, usando um processo estocástico para a parametrização do modelo. Os parâmetros foram escolhidos na literatura: Duffie (1999), Brigo and Afonsi (2005) and Andersen and Lung (1997).

Mostramos que os parâmetros do processo estocástico devem ser bem especificados e que a volatilidade tem de ser controlada. Os modelos de formas reduzidas mostram-se muito sensíveis e muito dependentes da amostra escolhida. A simulação da estimação de uma obrigação de cupom zero apresenta-nos que a média do preço simulada tem um declive decrescente quando o tempo aumenta. Este resultado sugere que a taxa de juro aumenta de acordo com o tempo segundo Mishkin (2010). Por conseguinte podemos observar que o modelo de forma reduzida incorpora um prémio de risco para premunir outrem contra o aumento de risco à medida que o tempo passa. As simulações de Monte-Carlo são concisas porque os intervalos de confiança estão muito perto da trajectória da média.

Concluimos que os modelos de forma reduzida insistem demasiado no facto de que a informação é unicamente estocástica. Como estipula Giesecke (2004), um modelo de informação incompleta pode ser uma melhor opção, pelo o menos em teoria, para superar o limite dos modelos de forma reduzida, que é o problema da generalização destes modelos para amostras diferentes.

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CHAPTER 1:

INTRODUCTION

Credit Risk has been the basis where both the 2007-2008 crisis and the European sovereign bond crisis emerged. Those crises occurred because of the loss of confidence which appears when asymmetric information takes place. This misinformation is the source of Credit Risk modeling in order to guard oneself against hypothetical loss. A vast number of models flourished in the literature (see Altman and *al.* (2004) for a good resume) since the pioneer work of Merton (1973). All this models of Credit Risk differ essentially on the information that can be perceived. For instance, structural models assume complete information as opposed to reduced-form models which suppose inaccessible information and incomplete information models which are built from partial information.

In order to understand how these models are constructed and which information is chosen to model it, we will pick one of them and simulate some measures of Credit Risk, namely the probability of default and the valuation of a zero coupon bond, following estimated parameters found in the literature (Duffie (1999), Brigo and Afonsi (2005) and Andersen and Lung (1997)).

The lack of information is the fundamental principle of Credit Risk so we will pick an example of a simulation of Credit Risk based in a reduced-form model provided in Duffie and Singleton (2003). This model implies that the event of a default cannot be predicted, and so, all the information of the model relies on a stochastic process. The default probability will be simulated through a stochastic process applying the Monte-Carlo method. We will find that the parameters of the stochastic process have to be well specified and calibrated in right proportion in order to have reliable results. The valuation of a reduced form model implies to define the relation between the default-free intensity, the risk-free interest rate and the recovery value. To give an illustration

we will simulate, by a Monte-Carlo method, a model of a zero coupon bond pricing before a default event, using a stochastic process for all parameters included in the model. No recovery and no correlation between parameters will be assumed. The results are in accordance with the theory (Mishkin (2010)), the price of the zero coupon bond is decreasing as the time goes by, showing that the reduced-form model can include a risk premium.

Both simulations (default probability and zero coupon bond pricing) are suffering from the decreasing velocity of the exponential, but their confidence interval is close to the mean estimator providing reliable results. In conclusion we specified that the reduced-form model might be too specific to be a good general model and it is too fitted to the chosen sample. An incomplete information model seems to be a more promising approach.

This dissertation is organized as follow; in the chapter two we will specify the definition of Credit Risk and the economic principles behind it, we will insist in the problem of information as the cornerstone of Credit Risk. The third chapter will introduce some notions in probability and statistics that we will use later to fully understand the reduced-form modeling. The fourth chapter will describe the three models used in the literature to model the default probability by specifying their flaws and their advantages and insisting on the information as the basis to set this kind of model. In Chapter 5 we will introduce the model that we will use to simulate the probability of default as a simple example. In Chapter 6 we will simulate the model specified previously with a one factor CIR model using the Monte-Carlo method. In Chapter 7 we will introduce a valuation model to price a zero coupon bond and we will stress on the possibility of correlations between various parameters. In Chapter 8 we will simulate a zero coupon bond with two CIR models one for the risk-free interest rate and another for the default-free intensity, using a Monte-Carlo method without any correlation and recovery rate. In Chapter 9 we will finish with the conclusion. In the appendix 1 we can find an explanation of the “*sde.sim*” package of the software R. In the appendix 2 we will present the R code for each simulation.

CHAPTER 2:

PRELIMINARIES IN CREDIT RISK MODELING

1. DEFINITION OF CREDIT RISK

Credit Risk mostly concerns financial intermediaries. Since indirect finance has been more important than direct finance. Banks and institutional intermediaries are the largest external sources of funds for business entities. So Credit Risk is a matter of concern for financial intermediaries (see Mishkin (2010)).

Credit Risk can be characterized in two ways, according to Duffie and Singleton (2003). One can define it as the risk that a borrower default or is unwilling to paid his obligations (the interest and the face value of the debt) to the lender. This is called a default risk.

Credit Risk can also be defined as the changes in the credit quality of a borrower. This is called the spread risk. If a borrower has a lower quality ranking we expect that he will be less able to pay off his running up debt. Therefore Credit Risk is characterized by two risks: default risk and spread risk.

As a consequence the debt issued by a borrower can be subject to default risk and changes in its credit quality modifying the spread of interest rate¹ underlying the issued debt. For instance, if a company just bears large losses, it will be less likely to honor the terms of its issued debt. Because lenders are risk adverse of a default and do not want to lose money, they are likely to charge a premium to bear an extra risk that the borrower can incur. The market traduces this risk in term of spread between a totally secured debt

¹ The interest rate can be defined as the yield to maturity which is the present value of future payments of a debt instrument with its value today, definition given by Mishkin (2010).

(like treasury bills) and a less secured debt (company debt for example). Hence the spread is only the difference between a default-free interest rate of a security and defaultable interest rate security.

Furthermore, changes on the quality of the debtor due to some loss means a greater risk of default and fewer expected return for the issued security of the borrowing company. The relative decrease in the expected return of this company compared with a risk-free security issued by another entity causes a decline to the demand for the security of the soured company. At the same time the company is becoming more risky, traducing that the price of the security issued by the company falls and its interest risk becomes greater relative to a risk-free security. This explains credit agencies existence; they assess the quality of entities and give some information to the lender about the risk of a default. Basically, Credit Risk assessment quantifies this risk in terms of probability, the probability of default, and in terms of losses for the lender, called the expected loss.

In short, as indicated above, the Credit Risk depends on two components: default risk and spread risk. The default risk is characterized by the event of default which in turn is defined by two components:

- The arrival risk that is the timing of the event conducing to the outline of the default probability.
- The magnitude of risk defined by the loss amount or/and the recovery value.

Hence, in one hand one can evaluate default risk through:

- The exposure at default/ recovery rates.
- The Default probability.

In the other hand one can evaluate spread risk by means of:

- Transition probabilities called also credit migration.

In this dissertation, we will not treat the spread risk and so the transition probabilities. Before proceeding ahead with Credit Risk models, to fully understand the notion of Credit Risk one can meditate upon fundamentals in Credit Risk. What are the basic economic principles underlying the Credit Risk?

2. FUNDAMENTALS ECONOMIC PRINCIPLES UNDERLYING CREDIT RISK

Credit Risk surges due to two economic concepts which are: the adverse selection and the moral hazard. The problem when a lender buys a debt security, like a commercial bank, is the missing information about the borrower ability to pay the interest payment and the face value of the issued debt when it matures. If we admit that the debtor cannot properly assess its ability to pay back his debt, asymmetric information takes place. This asymmetric information is the seed for adverse selection and moral hazard. The adverse selection arises from the fact that riskiest borrowers are more likely to ask for funds than safest ones. It is an easy point to catch. For bad quality borrowers issuing a debt even if the interest rate is very high is still profitable because returns are so high that it can completely offset the cost of borrowing. Of course the probability of default is bigger because the funds released have to be important to engage a borrower in a riskiest investment opportunity to get high expected returns. At the same time lenders will not fully profit of it because if the borrower is unable to pay back his obligations the lender losses money and if the borrower make a profitable investment the risk incurred by the lender is not reflecting the high returns that a borrower will make.

Another issue of incomplete information occurs ex post, when the borrower acquires the funds from the lender. The borrower can make riskiest investment in order to increase the expected value of returns. These changes in investment plan cannot be to the liking of the lender that support more risk than expected and cannot charge for it. This is called moral hazard.

Besides, we can indicate another problem caused by asymmetric information. It is the concentration of Credit Risk within groups, regions etc...Because some profitable opportunities can surge, for example different interest rate in two different branch of the same entities, creating adverse selection.

Anyway to get rid of asymmetric information in order to prevent adverse selection and moral hazard the lender has to collect information. That's what a financial intermediaries and credit agencies do. They practice monitoring and screen out

borrowers to assess their ability to pay back on their debt. The collection of information is essential for financial intermediaries for the purposes of lowering the Credit Risk.

Moreover, this necessity of information implies that the Credit Risk has to be quantified in order to be assessed.

Now that we have defined what involves Credit Risk we will proceed with its modeling. It will be helpful to make another specification before entering more precisely in Credit Risk modeling; therefore we will introduce briefly the notion of stochastic processes and diffusion processes, a framework to deal with the randomness of events present in Credit Risk.

CHAPTER 3:

PROBABILITY AND STATISTICAL NOTIONS

Modeling Credit Risk with a reduced-form model requires some probabilistic background in order to understand the specification of the model. The event of default will be modeled like an expected event over time which will incorporate uncertainty. Thus, we have to introduce succinctly some probability theory in order to fully comprehend the modeling.

1. PROBABILISTIC MATHEMATICAL MODEL

Let us define a random experiment \mathcal{E} as “an experiment whose outcome cannot be determined in advance” Ross (1996). So it presumes uncertainty. The probability theory will come up with tools which will give some mathematical foundation to analyze the hypothetical outcomes from the random experiment \mathcal{E} that are not known with certainty. So we call the set of all possible outcome of a random experiment a sample space, denoted Ω . Then, in probability theory an event is “a subset of a sample space, and is said to occur if the outcome of the experiment is an element of that subset” Ross (1996). In order to give a numerical outcome to the possibility of event we have to introduce the concept of a probability space where Ω is the sample space of all possible events of a random experiment \mathcal{E} and \mathcal{F} is the collection of all the events \mathcal{E} of Ω . A probability space follows 3 axioms:

For a particular event E of Ω :

Axiom 1: $0 \leq \mathbb{P}(E) \leq 1$

Axiom 2: $\mathbb{P}(\Omega) = 1$.

Axiom 3: For every event E_i and E_j where $i \neq j$ we have $E_i E_j = \phi$ (ϕ the null subset).

The sequence of event E_i with $i=1, 2, \dots, n$ follows:

$$\mathbb{P}(\bigcup_{i=1}^n E_i) = \sum_{i=1}^n \mathbb{P}(E_i).$$

Here we can see that \mathbb{P} is a set function with the following form: $\mathbb{P}(\cdot): \mathcal{F} \rightarrow [0, 1]$.

This probability space can be written as the triplet $(\Omega, \mathcal{F}, \mathbb{P})$. Since we define Ω as the sample space of all possible event of a random experiment \mathcal{E} , that for each event \mathcal{F} we have a function of the form $\mathbb{P}(\cdot): \mathcal{F} \rightarrow [0, 1]$ and if all the axioms of the space probability are respected we are in presence of a complete probability space, where $i \neq j, E_i \cap E_j = \phi$ and $E_1 \cup E_2 \cup \dots E_n = \bigcup E_i = \Omega$.

From now on we define a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a filtration $\{\mathcal{G}_t: t > 0\}$ satisfying the usual conditions of continuity which are:

For all t , \mathcal{G}_t contains all of the null sets of \mathcal{F} and for all t , $\mathcal{G}_t = \bigcap_{s>t} \mathcal{G}_s$, a property called right-continuity. A function $Z: [0, \infty) \rightarrow \mathbb{R}$ is left-continuous if, for all t , we have $Z_t = \lim_{s \uparrow t} Z_s$; the process has left limits if $Z_{t-} = \lim_{s \uparrow t} Z_s$ exists; and finally the process is right-continuous if $Z_t = \lim_{s \downarrow t} Z_s$.

In order to define what a stochastic process is we have to define the concept of random variable. A random variable X is defined as “a function that assigns a real value to each outcome in \mathcal{F} ” Ross (1996). So, now, instead of working with a subset \mathcal{F} from all universe (encompassing all possible events Ω) with a function set \mathbb{P} of \mathcal{F} that only gives number between 0 and 1 we work in the same subset \mathcal{F} but where we use another application X which associated, for every event \mathcal{F} of the sample space Ω , a real number defined in \mathbb{R} . Hence, here we can see that X is a set function of the form $\mathbb{P}(X = x) = \mathbb{P}(X^{-1}(x))$. Where $X^{-1}(x) = \{\varepsilon \in \mathcal{F}: X(\varepsilon) = x\}$, is a real number attached to the realization of the event ε belonging to the sample space Ω .

Now, from a random experiment ε , one can associate real numbers for the subset of the sample space. We can define for a continuous random variable X the distribution function which has the following general form: $F(x) = P(X \leq x)$ with $F(x)$ is a continuous function in \mathbb{R} and is derivable everywhere¹ in \mathbb{R} . $F(x)$ is a probability where $0 \leq F(x) \leq 1$, $\lim_{x \rightarrow -\infty} F(x) = 0$ and $\lim_{x \rightarrow +\infty} F(x) = 1$, and is always increasing. The density distribution function of a continuous random variable of X is defined, in any point where the density function $F(X)$ is derivable, as $F'(x) = f(x)$. Of course as $F(x)$ is always increasing, $f(x)$ is positive or null and $\int_{-\infty}^{+\infty} f(x)dx = 1$.

Some laws for random variables are very famous because they can model pretty well some event.

2. THE HAZARD RATE FUNCTION

In Credit Risk modeling we have a default event that is the same of modeling lifetime duration and for this a classical probability distribution used is the exponential law.

A random variable X has an exponential distribution with a parameter $\lambda > 0$ if

the distribution function have the form following:

$$F(x) = \begin{cases} 1 - e^{-\lambda x} & \text{when } x \geq 0 \\ 0 & \text{when } x < 0 \end{cases}$$

And the density function is defined as follows:

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{when } x \geq 0 \\ 0 & \text{when } x < 0 \end{cases}$$

The exponential distribution has the property to have no memory that is to say that:

$$P\{X > s + t | X > t\} = P\{X > s\} \text{ for } s, t \geq 0.$$

This property is important when we use the called hazard rate function. This is an essential function for the default probability modeling.

Let X be a continuous random variable X with a distribution function of F and a density probability function of f . An hazard function λ_t , is defined by:

¹ Except eventually in a finite number or in countable number of points.

$$\lambda_t = \frac{f(t)}{F(t)}$$

Let $f(t)$ be the density function that describes the probability distribution of T . Let $p(t) = P(T \geq t)$ be the survival function at time t . Plus, the hazard function $\lambda(t)$ specifies the instantaneous rate of default $T=t$ conditional upon survival to time t and is defined by limit for $dt \rightarrow 0$ of the following ratio:

$$\frac{P(t \leq T < t + dt | T \geq t)}{dt} = \frac{P(t \leq T < t + dt)}{P(T \geq t) * dt} = \frac{P(t) - P(t + dt)}{dt} * \frac{1}{P(t)}$$

$$\frac{P(t \leq T < t + dt | T \geq t)}{dt} = \frac{f(t)}{F(t)} dt = \lambda_t dt$$

Therefore the hazard rate function λ_t is called the probability intensity of an entity that survived until t and will fail in $t+dt$. Now, if we suppose that the survival probability $p(t)$ is strictly positive and is differentiable in t we can write that the distribution of T is specified by its hazard function and the continuous random variable T has an exponential distribution. Applying the memoryless property gives a constant $\lambda(t)$ which can be proved if we replace in the definition of the hazard rate function the adequate distribution function and the density probability function:

$$\lambda(t) = \frac{\lambda e^{-\lambda t}}{e^{-\lambda t}} = \lambda$$

The hazard rate function for the exponential distribution is constant.

Moreover the hazard rate function determines by itself the distribution function F of the random variable X :

By definition of the hazard rate function:

$$\lambda_t = \frac{f(t)}{F(t)} = \frac{-d/dt F(t)}{F(t)}$$

Ross (1996) demonstrates that at the end we will come up with this final equation:

$$F(t) = e^{-\int_0^t \lambda(t) dt}$$

Now that we have defined what is a random variable we can understand the definition of a stochastic process.

3. STOCHASTIC PROCESS

A stochastic process can be characterized as a series of random variables depending on time, $\{X(t)\} = \{X(t), t \in T\}$, Ross (1996). Basically a stochastic process is a dynamic set of random variables. If T is a countable set or a continuum set the stochastic process is respectively discrete or continuous. A realization of $X(t)$ is called a sample path. A continuous stochastic process has independent increments if for all $t_0 < t_1 < \dots < t_n$, the random variables $X(t_1) - X(t_0), X(t_2) - X(t_1), \dots, X(t_n) - X(t_{n-1})$ are independent.

A predictable process λ is a process where $\lambda: \Omega \times [0, \infty) \rightarrow \mathbb{R}$ is measurable of all left-continuous adapted processes.

A stochastic process is called an adapted process if it cannot foresee the future pass *ex ante*. We denote X as a adapted process if and only if X_T is known at time T for every realization and every t .

Some stochastic processes allow to model default probability in a Credit Risk modeling. Consequently we will present a particular stochastic process called a Poisson process.

So let set a counting process N , that is the total event that occurred up to time t , characterize by a increasing sequence $\{T_0, T_1, \dots\}$ of random variables valued in $[0, \infty]$, with an initial value $T_0 = 0$ and with $T_n < T_{n+1}$ whenever $T_n < \infty$. Let us set that: $N_t = n, t \in [T_n, T_{n+1})$, with $N_t = +\infty$ if $t \geq \lim_n T_n$. Suppose that this counting process is nonexplosive, that is, $\lim T_n < +\infty$ almost surely². This definition of T_n is the definition of a stopping time. In other words, for each T_i (with $i=0, 1, \dots, n$), the occurrence or non occurrence of the event $T=T_i$ depends only on the values of the counting process N .

And let λ a nonnegative predictable process such that, for all t , we have $\int_0^t \lambda_s ds < \infty$ almost surely. A Poisson process with a rate λ is a counting process N if:

² That signifies that the case where this happens have zero probability.

- $N(0) = 0$.
- The process has independent increments, that is, the events occurring during two disjoint times set are independent.
- In each interval with length t , all number of events are distributed with probability: $\mathbb{P}\{N(t+s) - N(s) = n\} = e^{-\lambda t} \frac{(\lambda t)^n}{n!}$.

Another essential feature of a Poisson process is the distribution of a sequence of interarrival times. If X_1 represents the time of the first event and X_n stands for the time between the event $(n-1)$ and n , the sequence $\{X_n, n \geq 1\}$ is defined as the sequence of interarrival times. Ross (1996) show that the sequence of interarrival times are independent and identically distributed exponential random variables having mean of $1/\lambda$.

Now let us talk about a non homogeneous Poisson process. A counting process $\{N(t) \geq 0\}$ is a non homogeneous Poisson process with a rate λ if

- $N(0) = 0$
- $N(t)$ has independent increments.
- $N(t) - N(s) \sim \text{Poisson}(\int_s^t \lambda(u) du)$.

If we denote $S_n = \sum_{k=1}^n X_k$ (the sum of the sequence of interarrival times to n), and if X_{n+1} is conditionally independent of all interarrival times given S_n , the distribution of X_{n+1} is:

$$\mathbb{P}[X_{n+1} > t | S_n = s] = e^{-\int_s^{s+t} \lambda(u) du}.$$

That will be an important result for modeling default probability. Another thing that we should define before entering in the Credit Risk modeling is the diffusion process.

4. DIFFUSION

A diffusion process is a continuous stochastic random variable with probability equal to one to have continuous sample path which have the memoryless Markov property, that is to say, the conditional distribution of the future state at time $t+1$, given the present

state t and all past states depends only on the present state and is independent of the past.

The most famous form of a diffusion process is the Brownian motion, denoted by $B(t)$ which satisfies the following properties:

- $B(0) = 0$.
- $\mathbb{E}[B(t)] = 0$ and $\text{Var}[B(t)] = \sigma^2 t$ more precisely $B(t) \sim N[0, \sigma^2 t]$.
- $B(t)$ has stationary, independent increments.

Diffusions are fully specified by their infinitesimal mean $\mu(x; t)$ and variance³ $\sigma^2(x; t)$. Besides, locally in space and time, $\mu(x; t)$ and $\sigma^2(x; t)$ are constant so all diffusions looks like, at least, as a Brownian motion with drift $X(t)$ where $X(t) = B(t) + \mu t$.

Hence, these local properties of $X(t)$ enables us to write that:

$$X(t + h) = X(t) + \mu(X(t), t)h + \sigma(X(t), t)[B(t + h) - B(t)] + o(h).$$

One can shows that to resolve this equation we can write the following stochastic differential equation (SDE) and solve it:

$$dX(t) = \mu(X(t), t)dt + \sigma(X(t), t)dB(t).$$

Therefore diffusions are solution for SDE. Depending on the infinitesimal parameters $\mu(x; t)$ and $\sigma^2(x; t)$ we can define an enormous variety of diffusion process.

5. MONTE-CARLO METHOD.

This is a method of simulation relying on random sampling to obtain numerical results. The first idea is to define the function that we want to simulate as a random variable. For instance, let set $X = f(x)$ where x are merely a realization of the function $f(\cdot)$ that can be of any pattern.

If we have a sequence of independent and identically distributed random variable $(X_i)_{i \geq 1}$ that are realizations of the function $f(x)$, we can set $X_1 = f(x_1, \dots, x_n)$

³ An infinitesimal mean and variance are the mean and the variance for a positive tiny increment of time: The infinitesimal mean is the form of: $\mu(x; t) = \lim_{\delta t \rightarrow 0} \frac{\mathbb{E}[X(t+\delta t) - X(t) | X(t)=x]}{\delta t}$ and the infinitesimal variance is the form of: $\sigma^2(x; t) = \lim_{\delta t \rightarrow 0} \frac{\mathbb{E}[(X(t+\delta t) - X(t))^2 | X(t)=x]}{\delta t}$.

, $X_2 = f(x_{n+1}, \dots, x_{2n}), \text{etc.}$ Therefore, those X_i are realizations of the random variable X and applying the law of large number we can write:

$$\lim_{n \rightarrow \infty} \frac{X_1 + X_2 + \dots + X_n}{n} = E(X)$$

Assuming that the expected value of X exists, ($E(|X|) < \infty$ and $E(X) = \frac{1}{N} \sum_{i=1}^N f(X_i)$).

Furthermore, the central limit theorem says that for $(X_i, i \geq 1)$ a sequence of the realization of the random variable X with $E(X^2) < +\infty$, and σ^2 the variance of X then $\frac{\sqrt{n}}{\sigma} \epsilon_n$ converge in law to $N(0; 1)$. Where $\epsilon_n = E(X) - \frac{1}{n} (X_1 + \dots + X_n)$ is the error.

The central limit theorem enables us to construct the confidence interval of the error, for a level of error of 5%:

$$\left[\frac{1}{n} \sum_{i=1}^n X_i - 1.96 \frac{\sigma}{\sqrt{n}}; \frac{1}{n} \sum_{i=1}^n X_i + 1.96 \frac{\sigma}{\sqrt{n}} \right]$$

Where n is the number of observations.

This is the method that we will apply later to do the simulation of the default intensity process.

Now that we have defined what involves the probabilistic notions that underline Credit Risk we will proceed with its modeling. First, we will focus on the default probability that can be, for instance, a triggering event or an expected event over time. But we will see that this definition of default probability is crucial and changes according to the model we choose.

CHAPTER 4:

CREDIT RISK MODELS OF DEFAULT PROBABILITY

Depending on how we define the default event different risk models are used. In fact, various ways of modeling Credit Risks have been developed to model default probability. The two main ways of modeling it is with structural-form models and reduced-form models. A third way has been the subject of study in recent years which combines both structural-forms and reduced-form way of modeling: they are known as incomplete information models. Besides, practical models have been developed by financial intermediaries and credit rating agencies relying on those models but using a Value-at-Risk (VaR) methodology¹. Some models are based on credit rating migration (for example the *CreditMetrics*® commercial model) where the forward probability (probability that a default occurs between times t and s) moves between one credits to another. Other VaR methodology models use a structural approach (like *KMV*® model) where the default process is intrinsically related with capital structure of a firm. The *CreditRisk +*® commercial model uses however a Poisson process to model forward probability which it can be viewed as a reduced-form model. For a comparison of the different model for VaR methodology models developed by some institutions one can refer to M. Crouchy, D. Galai, Robert Mark (2000).

1. STRUCTURAL-FORM MODELS

Primarily, structural-form models consider that the event of a default occurs based on balance sheet, when liabilities exceed assets in the balance sheet for example. Default is seen like a triggering event occurring over time. Structural-models defined default

¹ Typically the VaR methodology is a risk measurement used to estimate the loss level that will occur in a small fraction of the cases. For that purpose the loss probability distribution is set up, if the losses are larger than a threshold called VaR, the bank defaults. This methodology implies the knowledge of the probability distribution of the portfolio of a firm. In other words, the probability that a portfolio suffers losses larger than the sum of expected and unexpected losses is equal to a confidence level that is usually set at 99,9%. Mathematically we can write that VaR is:

$$P[L \leq VaR_{99,9}] = 99.9\%$$

within an economical conceptual framework. The base model is the Black-Scholes-Merton model (1973) which was extended in various ways afterwards. Typically this model implies that the default event occurs only at maturity date T of the debt if the liabilities of the firm are greater than the obligations. Furthermore, it is assumed that the information of the firm's asset value is completely observed by the model. This is a strong hypothesis that is not confirmed in reality. Merton (1973) assumes that the value of the firm follows a non negative diffusion process defined by:

$$dV(t) = rV(t)dt + \sigma V(t)dB(t)$$

The parameters r and σ are positive and are assumed to be constants. Merton (1973) simplifies the model when he considers only a unique zero coupon bond that matures at time T with a face value of 1, moreover the model only assumes that default can only happen at time T . In consequence, the probability of default happens for the event $V_t \leq 1$. At time T the lender (or the bondholder, the one in possession of the bond) will receive $\min\{V_t, B\}$ and therefore the value of the bond at time $t < T$ will be:

$$E_t[e^{-r(T-t)} \min\{V_T, B\}] = Be^{-r(T-t)} - P(t, V_T, B)$$

$P(t, V_T, B)$ is defined as the value at time t of a put option with strike B . The strike price equals the face value of the debt. And this formula is similar to the Black-Scholes (1973) formula:

$$Be^{-r(T-t)} - \Phi(-d_2) - V_T \Phi(-d_1)$$

With Φ is the cumulative distribution function of the standard normal distribution, $d_1 = [\log(A_t + (r + \frac{\sigma^2}{2})T)]/\sigma\sqrt{T}$, $d_2 = d_1 - \sigma\sqrt{T}$

This model has been extended in many ways afterwards, giving the possibility to these models to get rid of strong assumptions that are too restrictive to be true in practice. We can quote Leland and Toft (1996) who developed a model of optimal leverage and risky corporate bond prices for arbitrary debt maturity where the choice of debt maturity is view like a tradeoff between tax advantages, bankruptcy costs and agency costs. The value of the firm is still a diffusion process integrating a new constant that characterizes the rate of dividend paid to shareholders. This model assumes also constant interest rate.

Another interesting extension of the Merton model is allowing the default event to occur before the maturity of the debt. This assumption gives to those models a much more realistic facet. For this reason some authors entitle these models as second structural form-models. That is the case with the model developed by Black and Cox (1976). An interesting feature developed by Longstaff and Schwartz (1995) is to allow the interest rate to be stochastic using a Vasicek process². Besides, like Hull and White (1995) their model allows for a random time of default with a fixed length by modeling default as occurring at the first time when a default boundary is reached by a firm's value. We can quote as well a model built by Kim, Ramaswamy and Sundaresan (1993) that assumes that the value of the firm's assets follow a stochastic diffusion equation where the interest rate is a Cox-Ingersoll-Ross (CIR) model (specified in chapter 5). Plenty extensions of Merton model exist and characterize the structural models.

In a word, structural models, like Rogers (1999) noticed, show a clear link between theoretical economics assumption and defaults and they help to understand losses on default and correlation of default of different firms alike.

Nevertheless, these structural models suffer from three drawbacks as Altman, Resti and Sironi (2004) pointed out.

- The first drawback is the estimation issue of the parameters of the firm's asset value, since the market value of a firm is observable with extreme difficulty³, those parameters are unknown.
- Another drawback that arises in the structural models is the problem in assessing credit change of quality for risky defaultable debt. The credit downgrade represents a risk that has to be taking into account.
- The last inconvenience concerns the ability to predict the probability of default because the value of a firm is continuous over time without any jump process.

That is precisely what the reduced-form model are capable of, introducing a jump process in order that the default probability is not certainly known. In other words we do

² A Vasicek process, is a diffusion process with the following form: $dX_t = a(b - X_t)dt + \sigma dB_t$ with a the long term mean, b the speed of reversion, σ the instantaneous volatility and B_t a standard Brownian motion.

³ Except for the Black and Scholes formula when equity options are evaluated.

not specify any event of default. These three limitations of structural models are often pointed out to explain their poor empirical results (Jarrow and Protter (2004)).

2. REDUCED-FORM MODELS

Conversely of the structural-form models, reduced-form models consider that default occurs based on a stochastic process. Here, default is more likely an unexpected event occurring over time. No economical meaning is given to the default, that is, default is not influenced by the value of the firm. The reduced-form models are also known as the default intensity models. In these models default is not directly related with variability in a firm's asset value like structural models are, but an underlying relation exists, like the assumption of Lando (1998) implying that the intensity function depends on different state variables. Those models have been developed in order to resolve the drawbacks of structural models. Those models began with the work of Jarrow and Turnbull (1995). In this model the default time is assumed to be a stopping time at the first jump generated by a independent standard Poisson process N_t and an intensity process λ_t .

Generally, in reduced form model, λ_t always follows a non-negative process. The intensity process λ_t describes the conditional default rate which following Giesecke (2004) for a small Δt and $t < \tau$, the product $\lambda_t * \Delta t$ approximates the pricing probability that default occurs in the interval $(t, t + \Delta t]$, the probability of default prior to time T is:

$$P(\tau \leq T) = \mathbb{E}(e^{-\int_0^T \lambda_s ds})$$

The stopping time is formulated in a way that is totally inaccessible, that is to say, the modeler cannot predict when τ will occur. This differs from the structural-form model that allows prediction. The event of default can happen at each instant Δt and varies randomly over time. An exogenous stochastic variable drives probability of default and an interesting feature is that, at each time, default can occur so the probability of default is never null. In addition to take up the idea of Altman, Resti and Sironi (2004), default occurs when the Poisson random variable faces a discrete jump in level (or shift) given all available information. The reduced-form model requires less detailed knowledge

about the firm balance sheet than structural-form models. The value of the firm can be characterized by:

$$v(0, T) = \mathbb{E}([1_{\{\tau \leq T\}}\delta_t + 1_{\{\tau > T\}}1)e^{-\int_0^T r_s ds})$$

With $1_{\{\tau \leq T\}}$ a characteristic function that takes the value 1 when $\tau \leq T$ and 0 otherwise. And with $1_{\{\tau > T\}}$ a characteristic function that takes the value 1 when $\tau > T$ and 0 otherwise. r_s is the risk-free interest rate that derives from the arbitrage pricing principle and δ_t the recovery rate paid at time T.

Jarrow and Turnbull (1992) assumed δ and λ as constants. Many others authors contributed to the development of the reduced-form model, sometimes with different parameterization like the characterization of the recovery rate used in T, but mostly the difference between reduced-form models comes from different assumptions about the correlation between the default probability, the recovery rate and the interest rate allowing the distinction of one to another, according to Uhrig-Homburg (2002). One can see the work of Jarrow and Turnbull (1995) and Madan and Unal (1996), Duffie and Singleton (1999).

In conclusion reduced-form models can go beyond the structural-form limitations. But at the same time the reduced-form models do not possess an economic rationale when the model defines a default process; that increases the difficulty to assess a way to improve the performance of those models. This lack of specification however gives to the reduced-form model a good flexibility for its functional form. According to Rogers (1999) the flexibility can turn into a drawback because the reduced-form model is calibrated into empirical data to estimate parameters that fits the model which explain the better improvement in empirical studies. In other words, the model is built using a sample and might be a bad fitted model “out of the sample”. Anyway, the data collected are poor and incomplete to fit the reality which constitutes a problem because these models need to be test empirically with data to draw conclusions about their validity.

Jarrow and Protter (2004) see reduced and structural models rather alike but with different information assumption. So the real focus that one has to make is about the information known by the modeler. According to Jarrow and Protter (2004) structural

models assume complete and detail information about the firm like a firm manager and a reduced-form model possesses less information as the information observable to the market. Hence, one can switch to a structural model with default, acting like a predictable stopping time, to a hazard rate model with inaccessible stopping time. Finally, they conclude that in order to construct a pricing to the firm risky debt, the reduced-form is a better approach since we can observe continuously the firm's asset value.

Another type of model, more sophisticated, has been developed in recent years combining reduced-form models and structural-form models: the incomplete information models. For more detail see Duffie and Lando (2001) and Giesecke (2001).

3. INCOMPLETE INFORMATION MODELS

An important consideration to have in default models is the available information. Giesecke (2004) models the dynamic of investor information directly by the means of a filtration \mathbb{G} . Giesecke (2004) specifies a default model in two main parts:

- A default stopping time τ .
- A model filtration $\mathcal{F} \subseteq \mathcal{G}$.

So we can see that in a structural model, the stopping time is when the default event occurs before the maturity of the debt, before a fixed bound. Or, in earlier versions, at the maturity of the debt: when liabilities exceeded the assets value of the firm. And the filtration \mathcal{F} represents the information of investors concerning the balance sheet of a firm, the observed data to the model like firm assets etc. So the barrier default and the firm assets create the filtration \mathcal{F} that depends on the model definition of default. Here investors can fully observe the firm assets, so we are in a complete information model where the stopping time τ is predictable. According to Giesecke (2004) there is no short term Credit Risk in complete information model explaining the poor empirical behavior of structural models to fit market data.

In reduced-form models, investors are “blind”, they cannot observe the firm assets or the default barrier. Basically, they cannot anticipate the default and therefore the stopping time τ is unpredictable and the filtration \mathcal{F} is generated to model directly the

intensity. Tractable price securities can be identified and the short term credit spread is characterized by the intensity.

The incomplete information models are quite a mid-term between the two previous models. Investors have some information about the input of the model but this information is incomplete. In short, the model filtration \mathcal{F} can be generated by noisy or lagged observation of the asset value or the default barrier for instance. But the observations are incomplete and thus investors cannot predict the event of default hence the stopping time τ is totally inaccessible.

In incomplete information models we have the compensator A specified as follows:

$$A = A_{\min(t, \tau)} = \begin{cases} A_t, & t < \tau \\ A_\tau, & \text{otherwise} \end{cases}$$

The compensator has a definition of default like structural models. But then the default trend A_t can be parameterize directly through an intensity λ . Then the default trend takes the following form:

$$A_t = \int_0^t \lambda_s ds$$

Therefore the trend is like a cumulative default intensity because the default trend is nondecreasing with filtration \mathcal{F} and unpredictable process. If the trend stops at the stopping time τ it will be equal to the compensator A of the nondecreasing process that indicates default.

In conclusion, incomplete information models share the structural models definition of default and the inaccessible stopping time of reduced-form models.

One of the conclusions of the Giesecke (2004) is that “all incomplete information models lead to generalized reduced form security pricing formulae in terms of their trend”.

Here in order to model the default probability we will focus on reduced-form models. The default probability is seen like an unexpected event that is modeled by a stochastic process. Nowadays, with the globalization of financial market and the development of sophisticated financial derivatives the fact that default is an unexpected event is quite sustainable. It is such a complicated event to model that it appears tough to gather all the conditions under which a default can occur. The structural model is more orthodox, it impose a definition of default under which if the conditions are not met the default is

not occurring but this conditions can appear a little bit weak and default can occur without fulfilling all structural model default characterizations.

CHAPTER 5:

MODELING DEFAULT RISK

We will model the default probability by using a reduced-form model. We choose one of the many reduced-form models that exist and model the default probability. The choice was due to the stochastic processes used by Duffie and Singleton (2003).

Before entering with more details in the default intensity models let us define the meaning of forward default probabilities according to Duffie and Singleton (2003).

1. FORWARD PROBABILITY

The probability of default will be modeled conditionally on all available information, for each borrower. The arrival of new information might impact the probability of default.

We indicate $p(t)$ as the probability a firm can survive t years, in other words is the likelihood that a firm will not default during t years. Thus the probability of surviving to time s , given that the firm survived until t , knowing that s and t are independent and $s > t$ is:

$$p(s|t) = \frac{p(s \cap t)}{p(t)} = \frac{p(s)}{p(t)}$$

Hence the probability that a default occurs between times t and s , given survival to time t is $1 - p(s|t)$. This probability is called the forward default probability.

Let $f(t)$ be the density function that describes the probability distribution of T . Let $p(t) = P(T \geq t)$ be the survival function at time t . And the hazard function $\lambda(t)$ defined before (see chapter 3).

Now if we suppose that the survival probability $p(t)$ is strictly positive and is differentiable in t we can write that the distribution of T is specified by its hazard

function as well because the survivor function is determined by the hazard function. Therefore one can show that:

$$p(t) = e^{-\int_0^t f(u)du}$$

As s and t are independent and $s > t$ we can deduce:

$$p(s|t) = e^{-\int_t^s f(u)du}$$

That is the probability of survival to s given the survival in t . So $f(t)$ is the forward default rate and it can model the term structure of default risk. We can introduce the concept of default intensity according to Duffie (2002).

2. DOUBLY STOCHASTIC DEFAULT INTENSITY MODEL

Suppose that we have a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a filtration $\{\mathcal{G}_t: t > 0\}$.

Let N be a counting process, characterized by an increasing sequence $\{T_0, T_1, \dots\}$ of random variables valued in $[0, \infty]$. And let be λ a nonnegative predictable process such that, for all t , we have $\int_0^t \lambda_s ds < \infty$ almost surely.

The reduced-form models characterize default as the first arrival time τ of a Poisson process with some constant mean arrival rate which it is called intensity, denoted λ . Some properties of default intensity model derived from Poisson process, those properties are:

The probability of survival for t years is $p(t) = e^{-\lambda t}$, therefore the time to default is exponentially distributed.

The Poisson process is a counting process with a deterministic intensity process. Here we have supposed that the intensity of a Poisson process is constant so this default intensity model has independent arrival risk over time. Hence default time is said inaccessible, which means that is unpredictable. Here we cannot predict λ_t based on all of the information available up to time t but that do not include time t . Actually constant mean arrival does not hold in most of the cases. For this reason we can allow λ to vary over time in a deterministically way. So Duffie and Singleton (2003) conclude that if

the intensity process λ varies deterministically then it coincides with the forward default rate function f_x implying that survival is the only relevant information to default risk arriving over time.

Duffie and Singleton (2003) show that in the case of a deterministic continual variation in intensity we have:

$$p(t) = e^{-\int_0^t \lambda(s) ds}$$

with $\lambda(t)$ the intensity at time t . In a general framework the default intensity varies in a random way when additional information is added. So we will use a model with continuous variation in intensity. Hence, now we are using a non homogeneous Poisson process with a rate λ . Duffie and Singleton (2003) called this models Doubly Stochastic Default.

These models are conditional on the information given by the path of the intensity $\{\lambda(t): t \geq 0\}$, default arrives according to a Poisson arrival with this time-varying intensity. Let τ be defined as the stopping time. Duffie and Singleton (2003) show that the stopping time τ can be doubly stochastic with intensity λ . The doubly stochastic property implies that, for any time t , on the event that the default time τ is after t , the survival probability to a given future time with $\mathcal{G}_t = \{\lambda(s): 0 \leq s \leq t\}$ is: $p(t) = \mathbb{E}[P(\tau > s | \mathcal{G}_t)] = \mathbb{E}[e^{-\int_0^t \lambda(s) ds} | \mathcal{G}_t]$.

This is the sum of the sequence of interarrival times up to t (see chapter 3).

We can see that a double uncertainty arises: the default intensity process λ is random varying and conditional arrival default follows a Poisson process with time varying intensity. Or like Duffie and Singleton (2003) point out the doubly stochastic intuition is that the filtration \mathcal{F}_t contains enough information to reveal the intensity λ_t , but not enough information to reveal the events times of the counting process N . The intensity model is therefore specified but we still cannot compute the default probability. The problem that arises is that we do not know which form λ_t assumes over time. And without knowing it we cannot use the doubly stochastic model to calculate default probability. In other words, the question is: What is the parameterization of λ_t , how can we find it?

3. PARAMETERIZATION BY AFFINE MODELS

Many ways exist to parameterize the intensity model but a useful way can be using a state process X which is affine. An affine process X with some state space $D \subset \mathbb{R}^d$ is called a Markov process whose conditional characteristic function form, for any $u \in \mathbb{R}^d$ is,

$$E \left(e^{iu \cdot X(t)} \middle| X(s) \right) = e^{\varphi(t-s, iu) + \psi(t-s, iu) X(s)}$$

Furthermore Duffie and Kan (1996) present a variety of affine process with closed form solution for the default probability up to solution of an ordinary differential equation (ODE).

Duffie (2002) noted that for this formula it is sufficient that $\lambda_t = \Lambda(X_t)$ for some measurable $\Lambda: \mathbb{R}^d \rightarrow [0, \infty)$, where X in \mathbb{R}^d solves the following stochastic differential equation:

$$dX(t) = \mu(X(t), t)dt + \sigma(X(t), t)dB(t)$$

We suppose that this differential equation has a unique solution. This is a state process with the solutions X and $\mu(X(t), t)$ and $\sigma(X(t), t)$ being functions on the state space that is compatible which a unique solution.

Therefore, the survival probability calculation has the form of:

$p(t) = \mathbb{E}[P(\tau > s) | \mathcal{G}_t] = \mathbb{E} \left[e^{-\int_s^t \Lambda(X(u)) du} \middle| X(t) \right] = f(X(t), t)$. For the solution see Duffie (2002).

4. COX-INGERSOLL-ROSS MODEL (CIR)

Many affine processes exist but we choose a simple form of this affine process which is a process X solving the stochastic differential equation seem above referred and for which $\mu(X(t), t)$ and $\sigma(X(t), t)$ functions on the state space that are compatible with a unique solution. The CIR process is a Feller diffusion model (1951) named after Cox, Ingersoll, Ross (1985) to model term structure of interest rate. Many other papers confirm that the CIR model is a good candidate to evaluate the shape of the term

structure (see Andersen and Lund (1997), and also Brown and Schaefer (1994)). In addition the CIR model has been implemented in many famous literature modeling term structure of default risk one can referred to Duffie and Singleton (1999) and Brigo and Alfonsi (2005).

In order to simulate the probability of default we will use a simple parametric intensity model. We focus on the special case where there is one state variable ($d=1$ so state space $D \subset \mathbb{R}$), the volatility σ is constant and both this variable and the instantaneous default intensity process λ follow a “square-root” process describe as follow:

$$d\lambda_t = \kappa(\theta - \lambda_t)dt + \sigma\sqrt{\lambda_t}dB_t$$

Where κ is the long term mean reversion coefficient, θ is the long term mean of the process λ_t , σ is the volatility coefficient. Any CIR process is not negative. The parameters that satisfy that are $\kappa > 0, \sigma > 0$ and $\kappa \theta > 0$.

We can see that the restriction $\kappa \theta > 0$ implies that the process is positive if the initial value of $\lambda_0 = 0$. In fact at a date t when the process vanishes, when $\lambda_t = 0$, the process becomes deterministic locally since the Brownian motion part is null, only the drift remains $\kappa\theta dt > 0$. Both the deterministic part and stochastic part are affine functions of λ_t .

CHAPTER 6:

SIMULATION OF DEFAULT PROBABILITY

Now that we have defined a one state variable and that the intensity of the default probabilities follows a CIR model, we can simulate it. The CIR model is a stochastic differential equation (SDE) that its close form solution has been derived by Duffie and Singleton (2003). Therefore the close form solution of the CIR model gives the following form to the conditional survival probability $p(t, s)$:

$$p(t, s) = e^{\alpha(s-t) + \beta(s-t)\lambda(t)}$$

With α and β being time dependent coefficients. The only manner to simulate the conditional survival probability is knowing numerically α and β . But those parameters are unknown in reality. The only way to have an idea of their value is to use real data and to calibrate α and β in order to reflect the sample chosen closest to reality. No value of α and β is found in the literature. On the other hand we can still estimate directly the intensity λ_t in R using the package “*sde.sim*” (see appendix 1 for the description of the commands and how the simulation is run). The simulation of the CIR process needs to specify what is the starting value λ_0 and the three parameters, the mean θ , the reverting mean rate κ and the volatility σ .

The problem is how to find values for these parameters? Like for α and β these parameters have to be calibrated into real data coming from a sample. The problem is to find reliable data that permit to estimate the parameters as close possible of the reality. These parameters widely change across studies.

We can begin to test different parameters of CIR process (κ, σ, θ) chosen randomly in order to appreciate the reaction of the process of default λ_t . Then we use a Monte-Carlo method on the simulation outputs (see above chapter 3 for more detail). First we simulate various trajectories of the diffusion process independently and identically

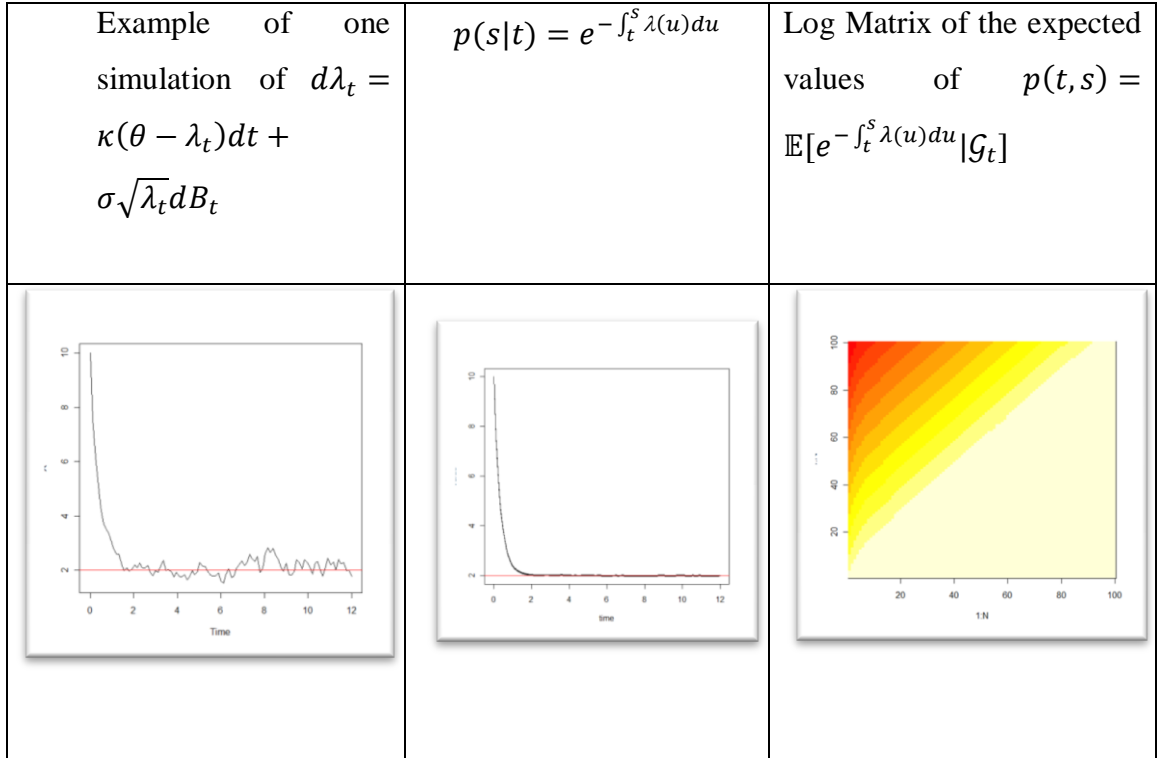
distributed, and then we use the idea of Monte-Carlo to assume that those trajectories that represent realizations of a random variable and we compute the expected value of the sum of those realizations (law of large number) and the confidence interval at 95% (central limit theorem). Hence we obtain the survival probability: $p(s|t) = e^{-\int_t^s \lambda(u)du}$,

which coincide in that case with the forward probability under the assumption of a deterministic variation of λ_t . We compute the logarithm of the forward probability because of the velocity of the exponential. Afterwards, under the assumption of random variation of λ_t we compute the doubly stochastic process (*survival prob*: $p(t, s) = \mathbb{E}[e^{-\int_t^s \lambda(u)du} | \mathcal{G}_t]$). We will compute the logarithm of the survival probability and the doubly stochastic survival probability because of the velocity of the exponential function. The confidence interval have the form of $\left[\frac{1}{n} \sum_{i=1}^n X_i - 1.96 \frac{\sigma}{\sqrt{n}}; \frac{1}{n} \sum_{i=1}^n X_i + 1.96 \frac{\sigma}{\sqrt{n}} \right]$ for a confidence level of 95%.

Notation: X_0 =initial value of the process, N =number of simulation steps, t_0 =time origin, T =horizon of simulation, M =number of simulations, θ = value of the parameters of the CIR process ($\kappa\theta, \kappa, \sigma$).

FIGURE1. SIMULATION OF THE DEFAULT MODEL

(see appendix 2: R code for more detail)



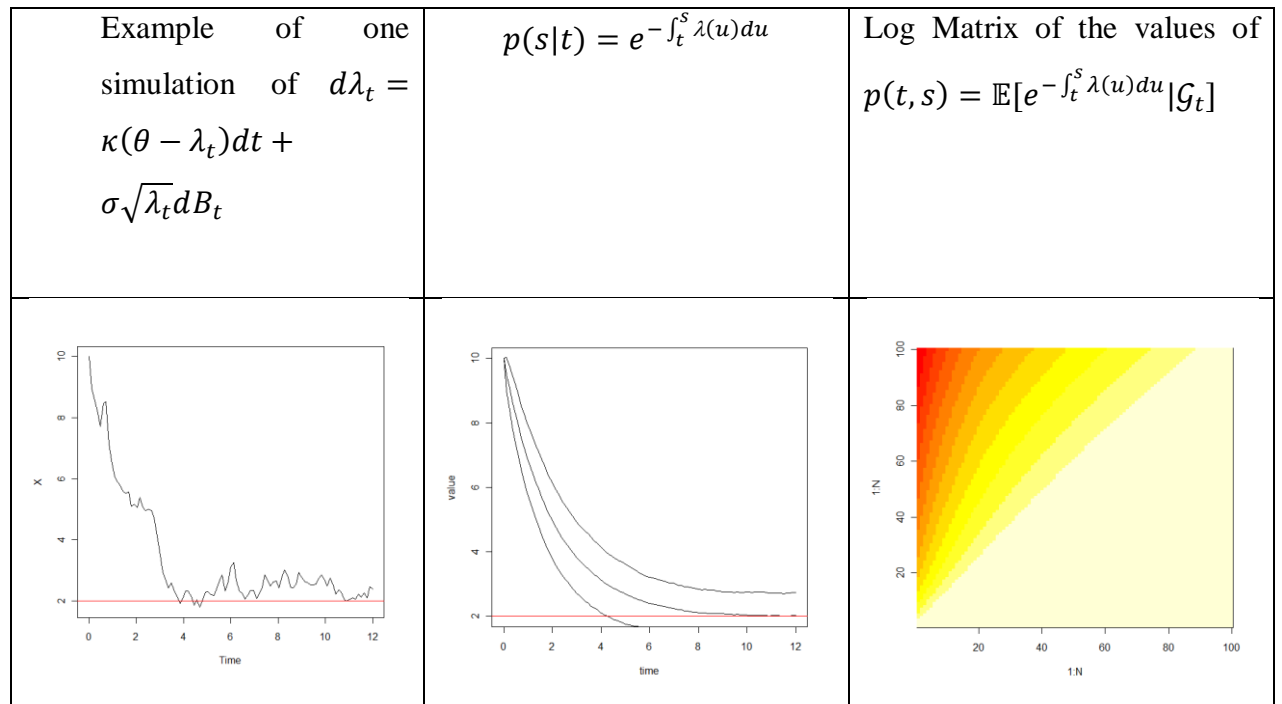
In figure 1, we can see that the single simulation of a the CIR process behaves like expected, the process starts at the initial value of 10 and then reaches rapidly, at time 2, its long term mean value (θ) of 2 basis point. The mean reversion rate is high ($\kappa=3$) explaining this quickness. We can see that the volatility is low ($\sigma=0.2$), the amplitude of fluctuation is limited. The survival probability with deterministic intensity λ_t behaves in the same way of the single realization of the diffusion process. This probability was simulated with 1000 realizations of the CIR process additioning to that the exponential function we can see a smooth downward splopping curve that reaches the long term mean at time 2. The confidence intervals are very close to the mean curve showing a good stability of the results. The last picture shows the logarithm of matrix of the doubly stochastic survival probability with a stochastic intensity λ_t , which is also a level curve. The effect of the exponential had been removed because the function decreased so quickly to its final state (the long run mean) that it was difficult to see the

transitional states. Even with the logarithm of the matrix of the function of doubly stochastic survival probability decreases fastly to the final state, the decrease in log terms is pratically proportional between the differents dots simulated. We can see that the dots of different states are increasing when the process is reaching its final state (which is the greater one), showing the convexity of the doubly stochastic function (inequality of Jensen preserved).

So we can be tempted to diminish the value of the mean reversion parameter (κ) to slow down the velocity of the process to reach the long term mean (θ). So in the next simulation we will hold constant all parameters except κ , and we will diminish significantly κ .

FIGURE 2. SIMULATION OF THE DEFAULT MODEL WITH SMALLER κ .

(see appendix 2: R code for more detail)

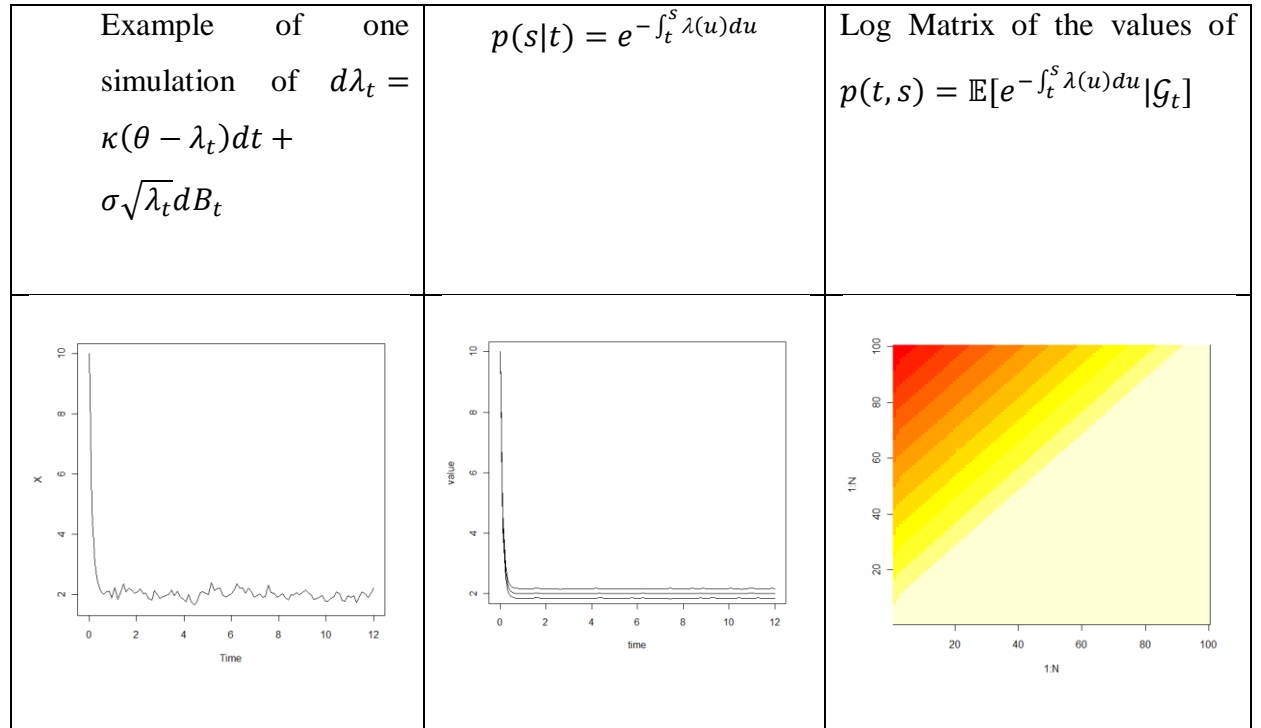


In figure 2, like expected the CIR process (one simulation) is decreasing more slowly to the long run mean ($\theta=2$). It is the same thing for the survival probability with constant λ_t . But the mean reverting is lower so the time period in which the long run mean is reached is higher, time period 4 instead of 2 for a higher κ (see the first figure). We can

see that for the log of the matrix of the doubly stochastic survival probability the higher values are passing to another state more slowly than lower values. The states near of the long run mean are greater due to the convexity of the doubly stochastic probability. The probability seems decreasing less rapidly than the one with an higher reverting mean rate.

FIGURE 3. SIMULATION OF THE DEFAULT MODEL WITH AN AUGMENTED κ .

(see appendix 2: R code for more detail)



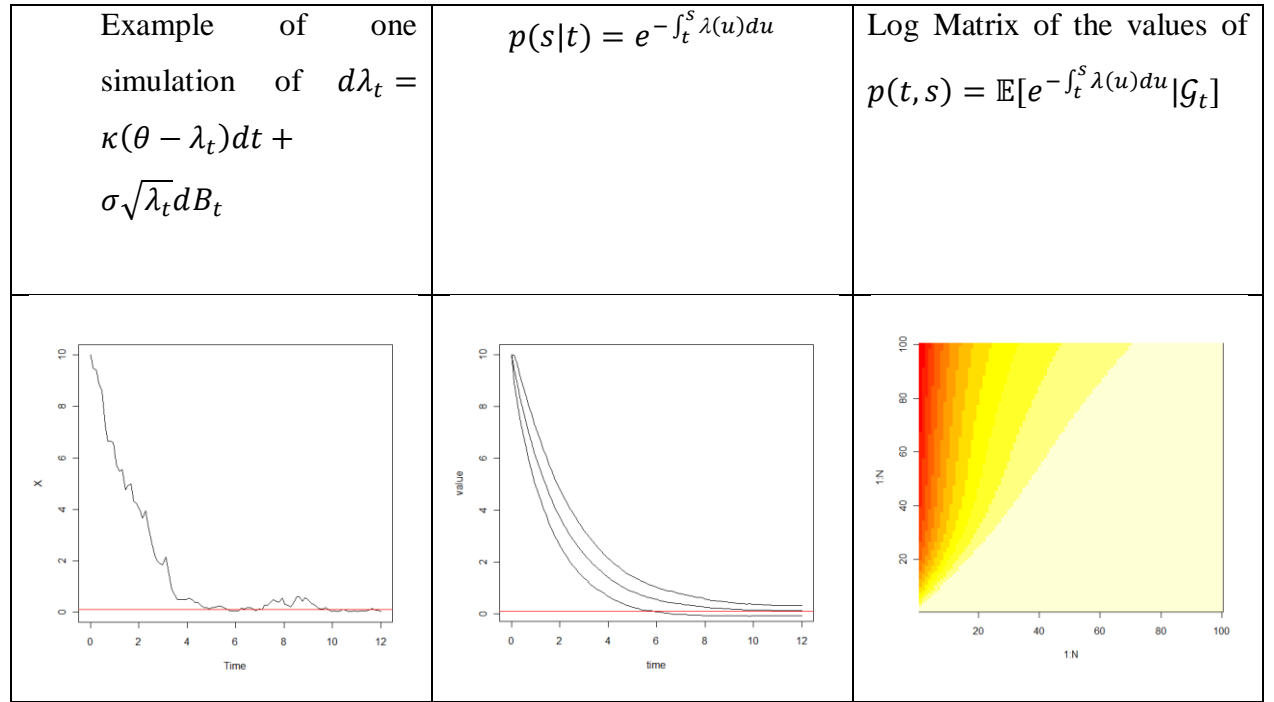
Like we expected the CIR simulation process decreases instantaneously to the long run mean (less than 1 period time). The stochastic part of the process is completely dominated by the deterministic part of the mean reversion rate (κ). The survival probability with a constant λ_t shows the same thing with very close confidence intervals confirming the dominance of the mean reversion rate. The logarithm of the matrix of the doubly stochastic process has basically the same form of the one with a lower mean reversion rate. So the deterministic part (mean reversion rate) or the stochastic part (browian motion) of the process does not have much influence when they dominate the other process, the form of doubly stochastic survival probability is still rapidly

decreasing to the final state. But here the doubly stochastic probability is more convex and the intermediate state are less higher.

We can be curious about what is the influence the long run mean (θ) can have on our modeling. What happens when we change the value of the long term mean?

FIGURE 4 : SIMULATION OF THE DEFAULT MODEL WITH A REDUCED θ .

(see appendix 2: R code for more detail)

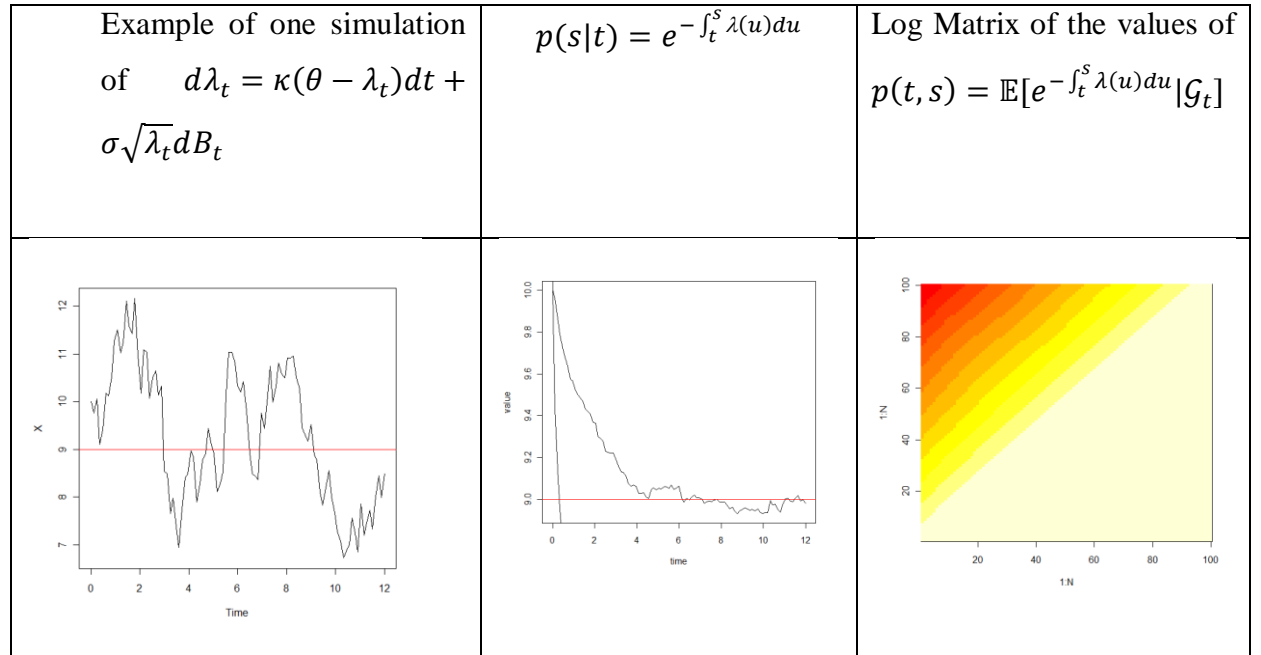


In figure 4, the long run mean is $\theta=0,1$, the single simulation of the CIR process and the 1000 realization of the survival probability with a constant λ_t decrease to 0,1 in more or less 5 period time for the single simulation of CIR process and 10 period time for the survival probability with constant λ_t . The log of the matrix of the doubly stochastic survival probability is quite different. For higher values the function takes more time to reach the final state (here the long run mean). The more we approach the final state the greater are the transitional states, so it takes more time for a final state to pass from a state to another. Therefore, the function seems more convex and it reaches more rapidly the final state than with a higher long run mean.

It should be interesting to see what happens when we increase the value of the long run mean nearer the inicial value, but without being higher. Because, of course, if the value is higher the process will be increasing and not decreasing.

FIGURE 5 : SIMULATION OF THE DEFAULT MODEL WITH AN INCREASE OF θ .

(see appendix 2: R code for more detail)



The simulation of the CIR process is fluctuating in turn of its long run mean ($\theta=9$), with a certain volatility as in figure 4. The simulation of the survival probability with a constant λ_t is decreasing quickly (it reaches the long term mean in fourth period instead of 10 with a long run mean lower or more distant of the inicial value). The log of the matrix of the doubly stochastic survival probability is decreasing to the final state rapidly but less rapidly and convex than with a long run mean more distant of the initial value.

Now we can see what looks like the simulation with some parameters present in the litterature: the first example comes from Duffie (1999) and the other comes from Brigo and Alfonsi (2005).

FIGURE 6. SIMULATION OF THE DEFAULT MODEL WITH DUFFIE'S (1999) PARAMETERS
CIR ESTIMATES.

(see appendix 2: R code for more detail)

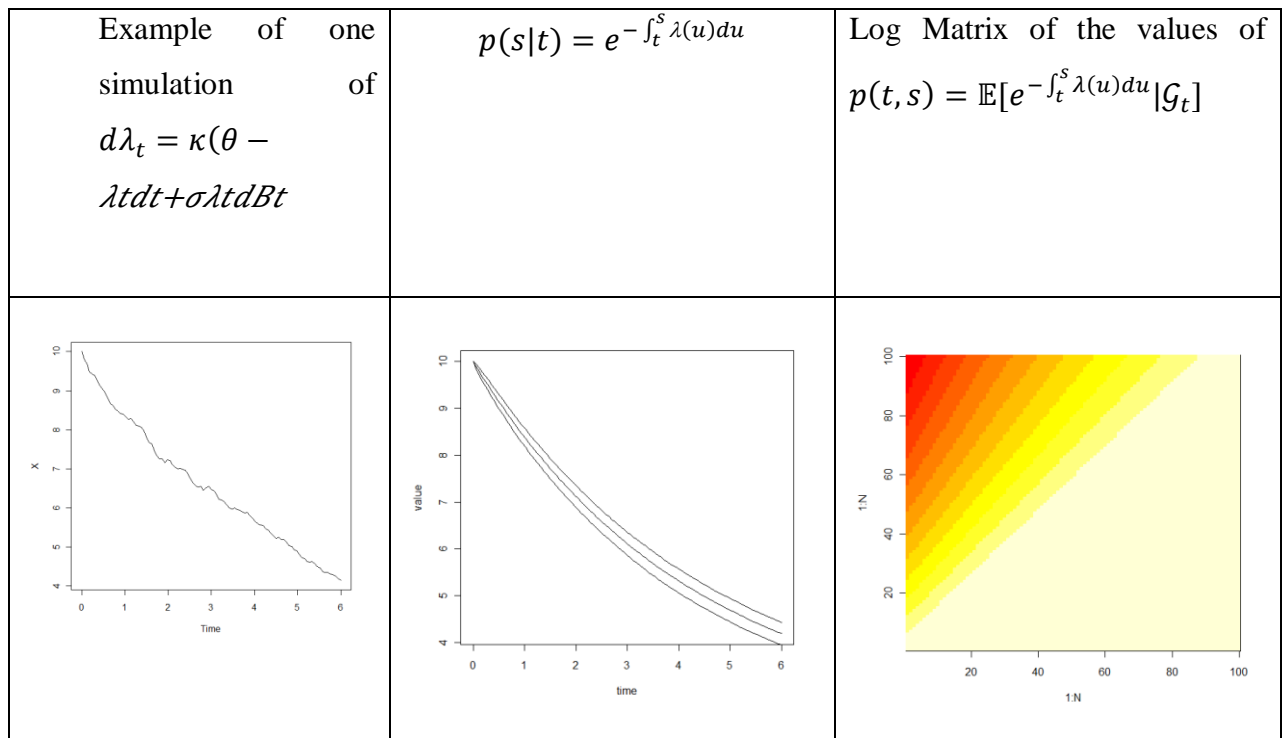
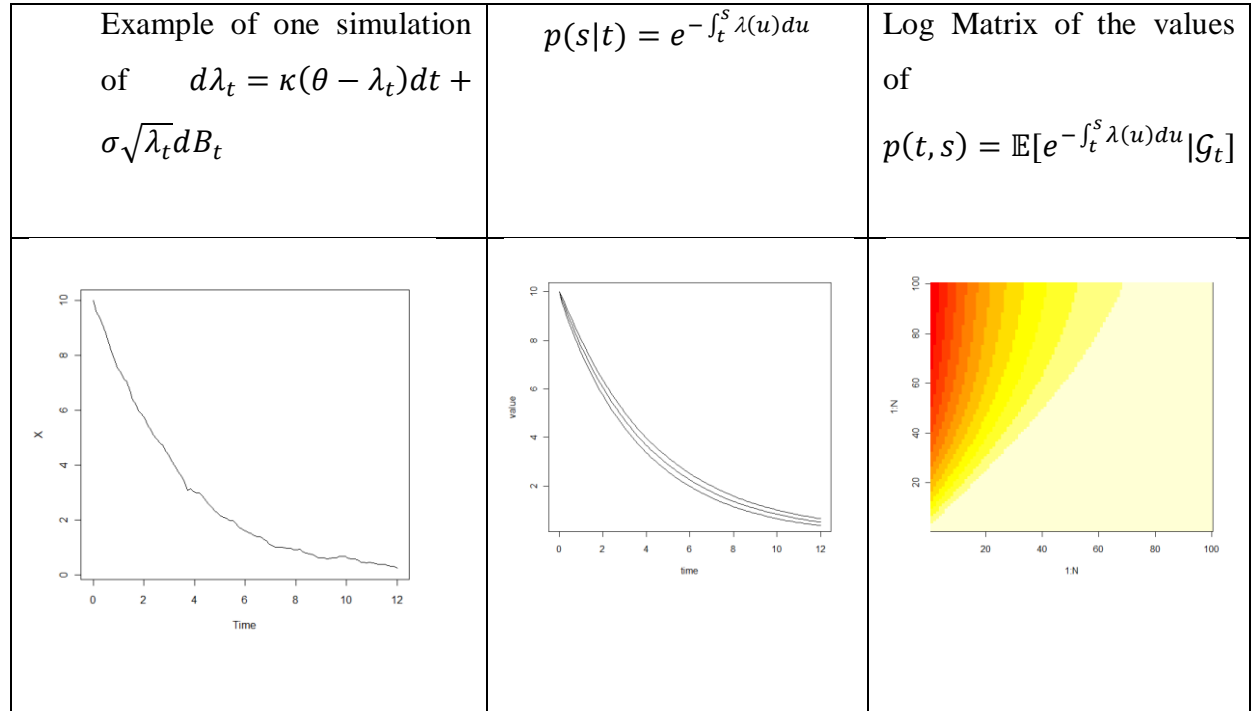


FIGURE 7. SIMULATION OF THE DEFAULT MODEL WITH BRIGO AND AFONSI'S (2005)
CIR PARAMETERS ESTIMATES.

(see appendix 2: R code for more detail)



In both situation the parameter of volatility is low (respectively $\sigma=0.074$ for figure 6 and $\sigma=0.1$ for figure 7). In both case the CIR process and the survival probability for a constant λ_t are decreasing for the long term mean (respectively $\theta=2.349$ for figure 6 and $\theta=0.005$ for figure 7), the mean reverting rate κ is quite close in the two simulations showing a similar behaviour. A particular difference exists in the logarithm of the matrix of the doubly stochastic survival probability (or level curve). In the paper of Duffie (1999) the survival probability have intermediate states much more longer than the one in the paper of Brigo and Alfonsi (2005). This is exactly what we have just seen above. The greater is the distance between the initial value and the long run mean the fastest the process reaches the final state (or the long run mean). The more we approach the final state the greater are the transitional states, so it takes more time for final state to pass from a state to another. Therefore the log of the probability of the doubly

stochastic survival probability seems more convex and reaches more rapidly the final state than with a long run mean higher.

In conclusion: The greater the reversion mean rate (κ) is, more rapidly the process reaches its long term mean (θ) and the transitional state is much more smaller. And vice versa.

The nearer the long run mean (θ) is to the initial value (λ_0), the more quickly the process decreases to the long run mean (θ) and the transitional state is much more smaller.

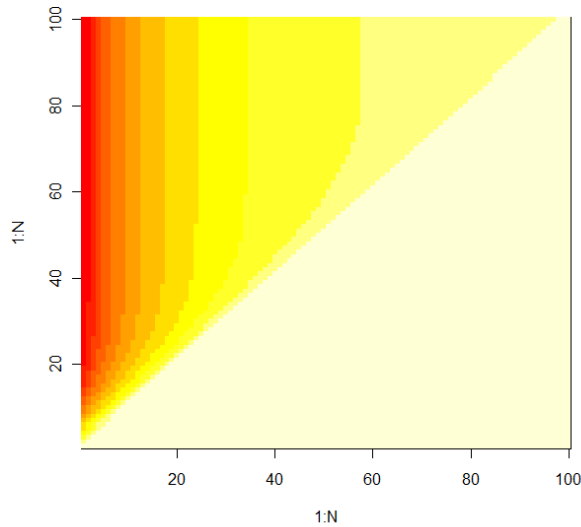
The greater the volatility (σ) the less stable is the process, the confidence intervals are bigger and if the process has a low mean reversion rate with some volatility the process is less stable too. And vice versa. Because when the mean reversion rate is lower the stochastic part of the CIR model is more dominant. The inverse is true.

So it appears difficult to model the doubly stochastic survival probability because of the high velocity in which the probability reaches its final state. Even with a low reversion mean (κ) and a more distant initial value regarding the long term mean (θ) and with $\lambda_0 > \theta$ (decreasing process), the doubly stochastic survival probability decreases too rapidly.

We can reach however an interesting result if we use the below parameters for simulation:

FIGURE 8. SIMULATION OF THE LOGARITHM OF THE MATRIX OF THE DOUBLY STOCHASTIC DEFAULT WITH LOW CIR PARAMETERS.

(see appendix 2: R code for more detail)



The idea is that in order to slow down the high velocity in which the process is going to, because of the exponential function, we can use a little value for each parameter.

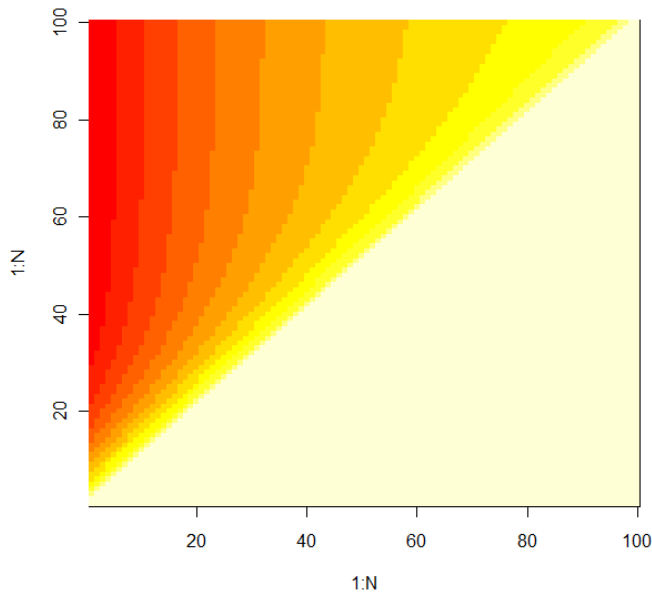
With those parameters we obtain the results displayed in figure 8, showing the logarithm of the matrix of the survival probability.

First, we notice that we have a wide range of intermediaries states clearly discriminated. Secondly we can easily see the convex form of all states (final states are bigger than initial states, which means that the function slowdown in final states).

But more surprisingly if we retrieve the logarithm and we compute the matrix of the survival probability, as one can see below (figure 9), we obtain something quite similar with intermediates states well defined and big.

FIGURE 9. SIMULATION OF THE MATRIX OF THE DOUBLY STOCHASTIC DEFAULT WITH LOW CIR PARAMETERS.

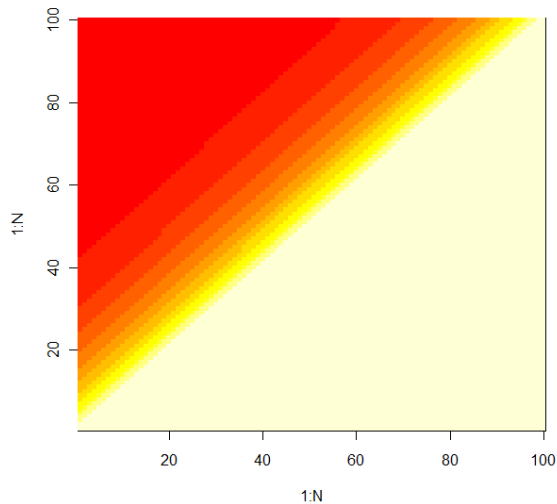
(see appendix 2: R code for more detail)



Nevertheless, this result is fallacious. The problem is that we got this simulation at the price of an uncontrolled and high volatility. If we compare the parameters of this simulation we can see that the volatility is much more bigger than the mean reversion rate (κ), about 10000 times bigger. And the velocity is 1000 bigger than the long run mean (θ). The simulation program R, using the comand *sde.sim* , alerts us that the process is not stationary which is normal because of the high volatility. To see this instability we can just divided by 10 the volatility and the result is quite different. The CIR process still continues to be not stationary, but the pattern of the function is quite different as one can see in figure 10.

FIGURE 10. SIMULATION WITH LOWER VOLATILITY OF THE LEVEL CURVE OF DOUBLY STOCHASTIC PROBABILITY.

(see appendix 2: R code for more detail)



We clearly have to keep certain proportionality between parameters if we want strong results. The problem that remains is that within those proportions the level curve basically shows only two states, one initial and one final, due to the velocity of the exponential function.

CHAPTER 7:

CREDIT RISK VALUATION

In the previous chapter we saw how to model the probability of default Credit Risk through a reduced form model conditional on current information. Nevertheless, characterizing the default event is not the main purposes of Credit Risk. Like we saw in chapter 1 the magnitude of risk is defined by the loss amount or/and the recovery value. To evaluate the event of risk we use 3 elements:

- The exposure at default/ recovery rates.
- The Default probability
- Transition probabilities also called credit migration

Here we will focus on the recovery rate, the short term interest rate and the risk neutral default free intensity that permits to quantify the Credit Risk in term of the recovery value. It is often more meaningful for financial institutions to define the risk of default in term of losses than barely model the event of default. Pricing derivatives, whose payoff is depending on a certain credit event, required an unambiguous definition of the credit event (like bankruptcy, downgrade, restructuring, merger, payment default). Here we continue to use reduce-form models that define exogenous and unpredictable credit event modeled stochastically. Still, pricing requires the notion of default-free probability that we will introduce below. An important setting in pricing modeling is the definition of the short term interest rate for the discounting that can be modeled to fit the historical data, the definition of the event of default can be made like the default model chapter above and the modeling of the recovery rate. Pricing, needs as well, to define a possible correlation of defaults between borrowers/risky assets. We just avoid the question of correlation between borrowers by assuming a good diversification of portfolio. At the same time one can think about what are the interdependences between the short-term

interest rate, the event of default and the recovery rate. However is precisely the different assumption about the correlation between these 3 parameters that differs from different pricing models. On conditioning in current information we determined the present probabilities of default events and it will be the same with the price of a security.

But first, pricing a security requires introducing two notions: risk neutral and actual probabilities.

1. RISK NEUTRAL AND ACTUAL PROBABILITIES

Risk neutral probabilities are the probability assessments under which the market value of a security is the expectation of the discounted present value of its cash flows, using the compounded short rate (Duffie and Singleton, 2003). In other words there are default probabilities implied from the credit market data. While the actual probabilities are the probabilities we just have seen in the previous chapter, which are the direct observations of default. We therefore introduced in the event of default timing conditional probability and conditional prices. An important question asked by many authors is that if, in accordance to Madan (1996), the ratio of price is of the same importance as the ratio of probabilities. In other words, if it exist a connection between the market price of a contingent zero coupon security and associated probability of the event of default.

By the same way taken the interest risk, Credit Risk is different from a portfolio to another because of the risk of default (when borrowers are unwilling or unable to pay back the interest payments and the face value of the issued bond). Investors are usually risk adverse for the time of default and the severity of the potential loss they can incur. In order to compensate the risk taken by lending money they want to earn an extra return to bear an additional risk. This is called a risk premium which indicates how much additional risk associated with default must take people to be willing to hold a defaultable bond. This explains the difference between risk neutral and actual probabilities.

Therefore, actual probabilities cannot reflect the true price of a security because risk premium is embedded in this probability. The true price of a bond can only emerge from

a risk neutral probability that is free from any premium. In other words the market price for a zero coupon bond does not reflect its probability. The ratio of market price does not equal the ratio of probability. Following Madan (1996) we consider a zero coupon bond contingent on some event X which the outcome we will know in a year. The sum of the price of the bond considering if the default occurs or not must be the value of a bond (expected discounted cash flows) $\frac{1}{1+r}$. With r the annual compounding spot interest rate for a one year term. These behave like a probability and the expected discounted cash flow for a zero coupon bond is the price $V(X) = \frac{Q(X)}{1+r}$.

Madan (1996) takes the rate return on the claim which is $1 + R(X)$ the ratio of the expected cash flow to the price:

$$1 + R(X) = \frac{P(X)}{Q(X)} (1 + r)$$

Hence the risk premium is the ratio of the probability to risk neutral probability $\frac{P(X)}{Q(X)}$.

It is straightforward to see that the risk premium is null if the ratio of the price and probabilities are equal. Like we have just said because of an existing risk premium the actual and risk neutral probabilities are not equal. In the same way if the risk premium is positive that means that actual probabilities are greater than risk neutral probabilities. And vice versa risk premium is negative when actual probabilities are lower than risk neutral probabilities.

Duffie and Singleton (2003) point out that the risk-neutral default-intensity process (noted $\{\lambda^*(t): t \geq 0\}$) might fluctuate randomly over time as new information comes into the market. Furthermore, they believe that no simple relationship might exist between actual default intensity λ and risk neutral probabilities in terms of level, random behaviour and degrees of persistence. One can try to parameterize the transformation between risk neutral-default intensity and actual default intensity. But this is beyond the scope of this work.

2. VALUATION MODELS: DIFFERENT ASSUMPTIONS

One of the pioneer works on reduced form model is the paper of Jarrow and Turnbull (1995). Their model was introduced in a discrete-time framework but they extended it to a continuous-time framework. One of the assumption is to model the event of default τ by an exponential distribution over $[0; \infty[$ and let the default intensity λ be constant. Therefore, the default time τ is identified as the first jump of a Poisson counting process with λ as parameter. The default-free forward rate is defined as a diffusion process. The forward rates (the actual path) have different characterization according to the value of t (prior bankruptcy $t < \tau$, bankruptcy $t = \tau$ and after bankruptcy if $t > \tau$).

Concerning the recovery rate it is assume fixed. Hence if we want to value a zero-coupon defaultable bonds the payoff ratio will be paying 1 at maturity T if there is no default and fixed recovery δ at T if the firm defaults before time T . Another important assumption from Jarrow and Turnbull (1995) is that only a unique equivalent martingale measure in arbitrage free exists and gives a proof for its existence in the model. Finally, Jarrow and Turnbull (1995) presume that the independence between the event of default and the spot interest rate- So it assumes that no interaction between the Credit Risk and market risk exists. That is a strong assumption that had been made to simplify the equations.

The work of Jarrow and Turnbull (1995) is a pioneer model that uses a unique martingale measure to price derivatives with risk neutral probabilities and a counting Poisson process to generate the event of default. After this model, a lot of extension have been made. The main differences lies in the assumption about how the interest free rate and the event of default and the recovery rate are modeled and whether correlation is allowed between this variables or not. For instance, Madal and Unal (1996) extend the previous reduced-form model by adding in the counting Poisson process to generate the default probability with a time varying intensity $\lambda(t)$ depending on the stock price (firm equity). This can be seen like what Lando (1998) introduced as the Cox process to model default probabilities. Madal and Unal (1996) distinguished from Turnbull (1995) by adding, as well, a stochastic recovery rate using a Beta distribution and payoff functions. But the independence between the default intensity and the default-free spot rate is still assumed and the stochastic process in which recovery is modeled is

independent of the default time. To deal with the correlation, one can refer to the work of Lando (1998) and Duffie and Singleton (1999). Those two models are interesting because they use the family of affine process to model the default time like we saw in Duffie and Singleton (2003) in the first chapter. As a reminder, the default intensity is modeled following a doubly stochastic intensity defined by a state variable X describing some Markov vector process. Both the default-free short rate and the intensity are modeled as they can express a linear combination of state variables. This allowed correlation between the default-free short rate and the default intensity through their affine combination of state variables. The main distinction of the two models is the recovery rate that is modeled differently but can be adapted in both models. That is the particularity of Lando (1998) and Duffie and Singleton model (1999) it's that they are general models that can be transform in many ways allowing to implement an simple correlation or no correlation between the default free interest rate, the default intensity and the recovery rate. For this purposes an illustration of valuation will follow Duffie and Singleton (2003 and 1999) specifications.

3. VALUATION MODELING

Valuation in finance is made by risk neutral probabilities “under which the market value of a security is the expectation of the discounted present value of its cash flows, using a compounded short rate for discounting. If we assume that a short-rate process exists such as is bounded $\int_0^t |r(u)| du < \infty$ for all t and that for any times $t > s$, an investment of one unit of account at time s reinvested continually in short-term lending until any time t after s , will yield a market value of $e^{\int_s^t r(u) du}$ ”, quoting Duffie and Singleton (2003). We have to remember that the interest rate is the present value of future payments of a debt instrument with its value today.

So if we set a probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ and a filtration $\{\mathcal{G}_t : t > 0\}$ we can set a martingale measure \mathbb{Q} (a probability distribution) based on discounting at the short rate r in order to compute present value. We can therefore write that for a zero coupon bond that pays 1 at T if there is no default and pays R at T if the firm defaults before time T can be modeled in a continuous time by:

$$\delta(t, T) = E_t^*[e^{\int_t^T r(u) du}]$$

Basically it is the same principle as modeling default intensity. These risk neutral probabilities only exist for no arbitrage condition as Harrison and Kreps (1979) in a complete financially market. But the real problem is that whether or not arbitrage opportunities exist because the actual and risk neutral probabilities will be different like we saw before and hence we need both probabilities.

In a general framework we can rewrite the above equation introducing a dummy variable. So when a zero coupon bond that pays F for $1_{\{\tau > T\}} = 1$ (no default by T) or zero when $\tau < T$ (*a default occurs*). At the same time Duffie and Singleton (2003) suppose as well that the magnitude of the payoff of a security can be subject to uncertainty. Therefore, we can add a dummy variable $1_{\{\tau \leq T\}}$ representing the event “default occurs before maturity”. Finally we can set the price of the survival-contingent security:

$$d_0(t, T) = E_t^* \left[e^{-\int_t^T r_s ds} 1_{\{\tau > T\}} \right]$$

Lando (1998) proves that if we assume that F , r , and $\lambda^{\mathbb{Q}}$ (define in a probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ and with a filtration $\{\mathcal{G}_t: t \geq 0\}$) and bounded $(\int_0^t |r(u)| du < \infty$ and $\int_0^t \lambda_s ds < \infty)$, under a local martingale \mathbb{Q} , then the stopping time τ is a doubly stochastic driven by a filtration $\{\mathcal{F}_t: t \geq 0\}$, with intensity process $\lambda^{\mathbb{Q}}$. Furthermore, if r is (\mathcal{F}_t) -adapted and F is \mathcal{F}_t -measurable, we can fix any $t < s$. Hence, for $t \geq \tau$, we have the zero coupon bond price is $d_0(t, T) = 0$, and for $t < \tau$, the zero coupon bond price (default has not occurred yet) is:

$$d_0(t, T) = E_t^* [e^{\int_t^T (r(u) + \lambda^*(u)) du}]$$

Where $e^{\int_t^T (r(u) + \lambda^*(u)) du}$ can be viewed has the discounted expected cash flow given the paths of r and λ^* . Here the assumption of no recovery is assumed. Under a martingale measure, Lando (1998) found an adjusted cumulative discount rate between t and T that encompasses the path of the default-free bond.

4. CORRELATION BETWEEN DEFAULT-FREE INTEREST RATE AND DEFAULT-FREE INTENSITY

That model implies that one can imposed correlation between the default-free interest rate and the default-free intensity. This model allows for dependence between $r(t)$ and $\lambda^*(t)$ and can be correlated through their joint dependence on $X(t)$. Furthermore, this can be viewed empirically that default rate is negatively correlated with the business cycle and obviously interest rate too (see Duffie and Singleton (2003)). Many forms exist in order to model this dependency to state variables. So we can use many risk factors in order to model $r(t)$ and $\lambda^*(t)$. One way used in Duffie and Singleton(1999) and Duffie (1999) ¹ is to model the risk factor through a diffusion process for each factor and imply that $r(t)$ and $\lambda^*(t)$ are affine process of the risk factors like:

$$r(t) = a_r(t) + b_r X(t)$$

$$\lambda^*(t) = a_{\lambda^*}(t) + b_{\lambda^*} X(t)$$

Where, according to Duffie and Singleton (2003), $a_r(t)$, b_r , a_{λ^*} and b_{λ^*} are deterministic and can be time dependent or not. Moreover, close form solutions of reduced model can be derived with those assumptions. If this correlation appears to be proved that means that variation on credit spread is not only due to change in credit quality but some others factors causes the spread.

As we defined earlier the spread is basically the difference between a default free interest rate and a non-default free interest rate. Or we can substitute interest rate about yield because we are assuming that the better way to measure the interest rate is the yield-to-maturity. We have already characterized the price form of zero-coupon corporate bond, hence we can define now the yield-to-maturity of such a bond for $t < T$:

$$Y_0(t, T) = - \frac{\log d_0(t, T)}{T - t}$$

If we apply the definition of a credit spread we got:

¹ Duffie and Singleton(1999) and Duffie (1999) use a one state variable modeled by a CIR process.

$$Spread(t, T) = Y(t, T) - Y_0(t, T) = -\frac{\log\left(d(t, T)/d_0(t, T)\right)}{T - t}$$

From the bond price we can see that if $\lambda^*(t) \equiv 0$ we stay with a default free bond, $d_0(t, T) = E_t^* \left[e^{\int_t^T r(u) du} \right]$. So the $\lambda^*(t)$ represent the risk premium that charges the lender to bear the risk of default. However, we can redefine the spread to be dependent of $q(t, T) = \mathbb{E}^* \left[e^{-\int_t^T \lambda^*(s) ds} \right]$ the conditional default probability:

$$Spread(t, T) = Y(t, T) - Y_0(t, T) = -\frac{\log(1 - q(t, T))}{T - t}$$

In conclusion the credit spread is supposed to depend on the default probability. So any change of default probability must have a direct impact to the spread. Like Zhang (2010) points out the movement of default rates and credit spreads both in level and in change, traducing an overestimation of the expected default rates most of the time. For him this is due to various unknown and known factors like the liquidity risk. An illiquid asset is most difficult to trade creating another risk to the lender that he can sell rapidly the security.

5. CORRELATION BETWEEN DEFAULT FREE INTENSITY AND ACTUAL DEFAULT INTENSITY

Another question that derives from reduced-form model using risk neutral probabilities deserves full attention. How can someone map the relationship between default free intensity and actual default intensity? According to Duffie and Singleton (2003), apart from a deterministic intensity as in the model of Jarrow and Turnbull (1995) the relation between default free intensity and actual default intensity has to be non linear because of the equation of Lando and the Jensen inequality². The idea of Duffie and Singleton (2003) is to correlate $\lambda^*(t)$ and $\lambda(t)$ through their joint dependence on a multivariable

² People prefer win for sure a certain average of amount X than uncertain average of amount X. If X is a random variable and f a convex function therefore Jensen shows that this following inequality is true: $f(\mathbb{E}[X]) \leq \mathbb{E}[f(X)]$.

state process $X(t)$, assuming for each intensity an affine dependence³. The risk factors can include both credit spread and default intensity and others relevant variables.

The important feature of valuation is modeling the magnitude of Credit Risk through the characterization of the recovery rate.

6. RECOVERY RATE

For the reduced form model the recovery rate is a capital assumption for pricing a security explaining the variety of reduced-form models that have been created. The reduced-form models assume a given expected fractional recovery conditionally on arrival of default. Two main approaches can be described with different view in the parameterization of the fraction: The Recovery of Face Value and the Recovery of Market Value.

The recovery of face value can be seen like a fraction of the face values of defaultable bonds that is given to bondholders when a firm defaults. This fraction comes from the liquidated assets of the defaultable firms. This recovery of face value assumes that the priority of bonds (senior, junior bonds) is followed but in practice is not the case.

According to Duffie and Singleton (2003) assuming independence between the default free-interest rate and the default-free intensity and a constant recovery-at-maturity fraction of the face value w^* : the price at time t of a defaultable zero-coupon bond with maturation at T is given by the following equation:

$$d(t, T) = (1 - w^*)d_0(t, T) + w^* \delta(t, u)$$

Where like we saw before the default free interest rate is $\delta(t, T) = E_t^*[e^{\int_t^T r(u)du}]$,

$d_0(t, T) = E_t^*[e^{\int_t^T (r(u) + \lambda^*(u))du}]$ is the price of a zero-coupon bond without recovery rate like we see above.

This is a simplest assumption that permits to model the recovery rate quite easily. Relaxing the recovery-at-maturity Duffie and Singleton (2003) shows that we can write

³ In the same way as the correlation between the default free interest rate and the default free intensity has been constructed above.

the price at time t of a defaultable zero-coupon bond with maturation at T is given by the following equation:

$$d(t, T) = d_0(t, T) + w \int_t^T \delta(t, u) \pi^*(t, u) du$$

Where like we saw before the default free interest rate is $\delta(t, T) = E_t^*[e^{\int_t^T r(u) du}]$,

$d_0(t, T) = E_t^*[e^{\int_t^T (r(u) + \lambda^*(u)) du}]$ is the price of a zero-coupon bond without recovery rate like we see above. And finally $\pi^*(t, u)$ is the risk neutral density of the default time conditionally on the available information at t . It can be show that $\pi^*(t, u)$ can be characterized by the following equation:

$$\pi^*(t, u) = E_t^*[e^{\int_t^u \lambda^*(s) ds} \lambda^*(u)]$$

One can derive close form solution if the default free intensity follows an affine type distribution $\lambda^*(t) = a_{\lambda^*}(t) + b_{\lambda^*} X(t)$.

We can relax the assumption of independence between the default free short interest rate and the default free intensity, in this case we can rewrite the price of the zero coupon bond according to Duffie and Singleton (2003):

$$d(t, T) = d_0(t, T) + w \int_t^T (E^*[e^{-\int_t^u [r(s) + \lambda^*(s)] ds}]) du$$

Close form solution can still be derived if the default-free intensity follows an affine type distribution $\lambda^*(t) = a_{\lambda^*}(t) + b_{\lambda^*} X(t)$.

Another extension of the recovery of face value is assuming a stochastic recovery fraction of the face value $w(u)$. The idea is that prior default a bond can be priced if we model the expected recovery conditionally of all information that occurred only before the stopping time τ . So the conditional expected recovery can be noted $E(W|\mathcal{F}_{t-})$ and do not include information on τ . Let $w(u) = E(W|\mathcal{F}_{t-})$ we can rewrite de pricing bond with stochastic recovery:

$$d(t, T) = d_0(t, T) + w \int_t^T E^*[e^{-\int_t^u [r(s) + \lambda^*(s)] ds} \lambda^*(u) w(u)] du$$

Duffie and Singleton (2003) observe that close form solution can be derived under certain affine specifications of the risk-neutral expected recovery rate $w(u)$ as a function of the underlying risk factors.

The other approach to model the fractional recovery rate is the Recovery of Market Value.

The idea, developed by Duffie and Singleton (1999), is that conditioning on all available information up to t but excluding the time t and for each time t , a risk neutral mean fraction L_t of market value is lost if default time occurs at time t . So here the fractional recovery is not constant and the risk neutral conditional expected recovery rate for a loss of market value if default occurs is $s_t = \lambda_t^* L_t$. Assuming recovery of market value Duffie and Singleton (1999) show that the price of a zero coupon bond at any time t before default is:

$$d(t, T) = \mathbb{E}_t^*[e^{-\int_t^T [r(u) + s(u)] du}]$$

An interesting feature is that according to Duffie and Singleton (1999) one can allow correlation between default-free intensity and default-free interest rate through a default-adjusted short rate $R = r + s$. So the price of a zero coupon bond at any time t before default can be written:

$$d(t, T) = \mathbb{E}_t^*[e^{-\int_t^T [R(u)] du}]$$

Moreover a liquidity effect l can be introduced in the default-adjusted short rates $R = r + s + l$. This manner to model R permits a closer modeling of the theory of liquidity premium theory.

CHAPTER 8:

SIMULATION OF VALUATION MODEL

Like we saw in the last chapter, evaluating Credit Risk by reduced-form model is much more complicated. The simulation of the valuation with reduced-form model suppose to dispose of a certain data none always accessible, much of the time none really reliable. The method to find the parameterization of default-free rate interest rate and the default-free intensity is troublesome and needs particular methodology that is avoided here. Another issue is which financial derivative do we want to price? Because the model will be completely different according to the derivatives that one chooses.

The main purpose of this chapter is just to perform, as we did in chapter 6, a simple simulation of the valuation model that we specified in chapter 7, without any recovery value or any correlation between default-free interest rate and default-free intensity. We can fix any $t < s$. Hence the zero coupon bond price is:

$$d_0(t, T) = \begin{cases} E_t^* \left[e^{\int_t^T (r(u) + \lambda^*(u)) du} \right] & \text{when } t < \tau \\ 0 & \text{when } t \geq \tau \end{cases}$$

We assume that both $r(u)$ and $\lambda^*(u)$ follow a different CIR process $(\kappa\theta, \kappa, \sigma)$. The method of simulation is the same as in chapter 6, using a Monte-Carlo simulation of 1000 realizations for the logarithm discounted expected cash flow given the paths of r and λ^* , $(\int_t^T (r(u) + \lambda^*(u)) du)$. And then we will simulate the price of a zero coupon bond when default has not occurred yet, $t < \tau$. We use a CIR process with the same parameters of Duffie (1999) for the default-free intensity and the same parameters of CIR used in Andersen and Lung ¹(1997) for the risk-free interest rate. This parameters has been chosen to give a notion about their possible values. Another specification for this simulation is that the discounted expected cash flow is in a logarithm form as well

¹ Used a continuous CIR process with one factor model, estimate by Efficient Method of Moments, for three month treasury bills.

as the price of coupon bond because of the high velocity of the exponential function. The confidence interval have the form of $\left[\frac{1}{n}\sum_{i=1}^n X_i - 1.96\frac{\sigma}{\sqrt{n}}; \frac{1}{n}\sum_{i=1}^n X_i + 1.96\frac{\sigma}{\sqrt{n}}\right]$ for a confidence level of 95%.

FIGURE 11. SIMULATION OF ONE PATH OF THE CIR PROCESS FOR RISK-FREE INTEREST RATE.

(see appendix 2: R code for more detail)

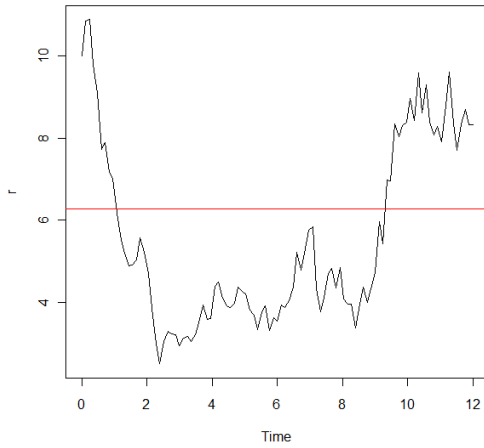


Figure 11, shows the simulation of a CIR process for the risk-free interest rate turns into the the mean $\theta=6.279$. But because of the high volatility of the CIR process ($\sigma = 0.67$) comparatively too the other parameters the mean reversion parameter ($\kappa = 0.082$) have some difficulty to bring the path to the long term mean θ . This is in accordance with the usual volatility of interest rate despite the fact that this parameters have been calibrated to tree-month treasury bills. The initial value is chosen at random.

FIGURE 12. SIMULATION OF THE LOGARITHM DISCOUNTED EXPECTED CASH FLOW.

(see appendix 2: R code for more detail)

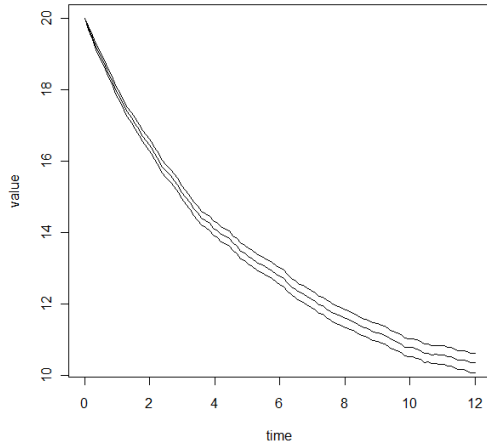


Figure 12, simulates the logarithm of the discounted expected cash flow². The sum of the two independent CIR process are decreasing gradually reflecting that over time a discounted cash flow decrease because of the time value of money³ and the risk premium. With the notion of the time value of money the price of the zero coupon bond falls naturally. Plus, the default-free intensity can be seen like we saw in chapter 7 as a risk primum that drives the risk-free interest rate to increase and so the price of the zero coupon bond to fall. We can see that the confidence interval is very close to the estimated mean of the simulation, hence the estimated mean is a great estimator of the true mean. The simulation is very precise.

² As a remainder the discounted cash flow is defined, $DCF = \frac{CF_1}{(1+r)^1} + \dots + \frac{CF_n}{(1+r)^n}$. (CF=cash flow). We can see the decreasing relation between the discounted cash flow and the interest rate.

³ The money worth less in the future that in the present days because of inflation and because of the fact that holding money does not give any interest rate so it is better to invest a rent. (See the definition of yield in chapter 7 for a better understanding).

FIGURE 13. SIMULATION OF THE LOGARITHM OF THE PRICE OF A ZERO COUPON BOND BEFORE DEFAULT.

(see appendix 2: R code for more detail)

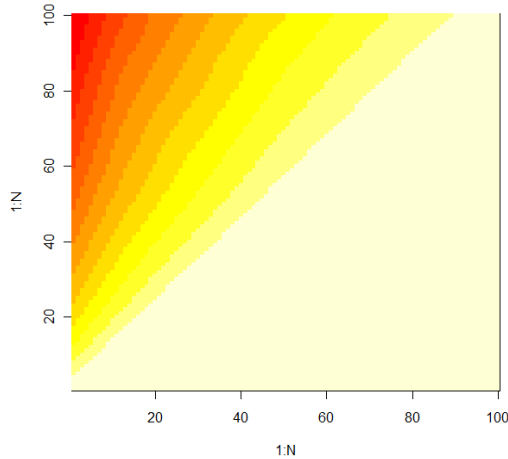


Figure 13, shows the level curve (in logarithm terms) of the price of a zero coupon bond before default, because after default the price is null. As we can see the level curve is decreasing to the final state (the beige zone). Even in logarithm terms the decrease of the level curve is sharp. The states are becoming greater as we are getting nearer to the final state, reflecting the convexity of this level curve. The states are not proportional representing the non linearity of the level curve. So in terms of concrete interpretation we see that the logarithm of the price of the zero coupon bond diminish until the stopping time τ where it becomes null. That is a logical result since the price is diminish as time pass inducing that the interest rate is increasing because of the negative relationship between the two variable. It is normal that in long term the interest rate is greater than the interest rate in short term⁴, see Mishkin (2010), because of the risk premium embedded in a possible default of the borrower characterize in this model by the default-free intensity λ^* .

⁴ But is not always the case, the term structure of interest rate (curve of interest rate at different maturity) can be downward sloping sometimes.

CHAPTER 9:

CONCLUSION

To conclude, we can stress that the principal element that is the cornerstone of Credit Risk and its modeling is the information that one can perceive. This information can be complete (structural-form models), partial (incomplete information models) or not available (reduced-form models). This perceived information defined the methodology that one can apply to model Credit Risk. Everything lies on this available information or not. And that is the very fundamental economic notion of Credit Risk. We can say that information is the essence, is what makes Credit Risk to be what it is. Because investors do not know the future for sure, they can lose money and they want to be covered by this risk because of the principle of risk aversion. That lead us to what Giesecke (2004) explain so well, to model default probability we have to consider when is the stopping time, the time where default occurs, and how investors, though what Giesecke (2004) called a filtration, can see about the viability of some firm or institution. This will define completely the default probability. In consequence, if Credit Risk is a problem of missing information it can appears utterly phantasmagorical to use a model which assumes that complete information exists (structural models). With all this embedded information it appears that the information is not observable and that is why we choose the reduced-form model.

The modeling of reduced-form that we have exposed and simulated, whether default probability or magnitude of default through pricing modeling, was reproduced in a very simple manner of course because of the difficulty in finding available data, constructing complex estimates etc. We showed that stochastic processes have to be well calibrated because the result can be very difficult to interpret (three dimension graph and the use of the exponential). The parameters are widely different across studies. But we reproduce the way the default probability and the price of a zero coupon bond are set. In default probability we have seen that the CIR model has a stochastic model need to be well parameterized to be meaningful. A change can gives some different results. And the volatility has to be controlled to limit the effect of the stochastic part of the process.

Pricing appears a much more complicated issue. It depends on the relationship of risk-free interest rate, default-free intensity and recovery value. This depends on the information that we want to instill to the model. The simulations by the Monte-Carlo method are precise, the confidence intervals are close to the simulation path. The results are in accordance with the theory. Despite the fact that it was a simple reproduction of a Credit Risk model we could appreciate a large bunch of literature with many models, which are only adapted to specific data.

Reduced-form models fit the data with parameters provided by stochastic phenomena but their high specificity to the sample cannot build a general model. Therefore, incomplete information model appears quite a promising idea, with an unpredictable stopping time and a filtration not only embedded in a stochastic intensity but in some parameters of balance sheet data. There is a promising future to coming models will be set and try to get closer to the reality of the whole phenomena of Credit Risk.

APPENDIX 1:

“SDE.SIM” PACKAGE

The sde.sim package permits through simulation different paths of solution to generic stochastic differential equations. The sde.sim package simulates diverse Stochastic Differential Equation. Here we chose the CIR process to simulate both the default event and the default-free interest rate.

$$d\lambda_t = \kappa(\theta - \lambda_t)dt + \sigma\sqrt{\lambda_t}dB_t.$$

The sde.sim package use different discretization to run the simulation. One that we have chosen is the Euler scheme for the discretization.

First, in order to run the Euler scheme someone have to specify an initial deterministic value at t_0 , $X_{t_0} = X_0$, the discretization Δ of the interval $[t_0; T]$. The Euler approximation of X is a continuous stochastic process satisfying the iterative scheme:

$$Y_{i+1} = Y_i + b(t_i, Y_i)(t_{i+1} - t_i) + \sigma(t_i, Y_i)(B_{i+1} - B_i)$$

So for a CIR process the iterative schemes are:

$$Y_{i+1} = Y_i + \kappa(\theta - Y_i)(t_{i+1} - t_i) + \sigma\sqrt{Y_i}(B_{i+1} - B_i)$$

This simulation is only possible only at t_i . Technically, to simulate the times in the interval $[t_i; t_{i+1}]$, a linear interpolation can be set up, defined by $Y_i(t) = Y_i$ for all $i \in \{0, \dots, n-1\}$ and by:

$$Y(t) = Y_i + \frac{t-t_i}{t_{i+1}-t_i} (Y_{i+1} - Y_i) \text{ for all } t \in [t_i, t_{i+1}).$$

If we do not specified the coefficient of sigma it is assume to be unitary. We have to indicate the initial value X_0 , the interval $[t_0; T]$, the delta steps (if is not specified we it use $\Delta = (T - t_0)/N$), the new simulated values of the process to be generated N , the

number of trajectories of the same process M . If $M=1$ the output is a single time series, if $M \geq 2$ the output will be a multi-dimensional time series.

APPENDIX 2:

R CODE

Figure 1. simulation of the default model

Example of one simulation of CIR process.

```
X0=10
N=100
t0=0
T=12
M=1
theta1=6
theta2=3
theta3=0.5
library(sde)
sde.sim(X0=X0, N=N, M=M, t0=t0, T=T, theta=c(theta1, theta2, theta3),
model="CIR") -> X
plot(X)
abline(h=2, col = "red")
```

The survival probability with deterministic intensity λ_t

```
X0=10
N=100
t0=0
T=12
M=1000
theta1=6
theta2=3
theta3=0.5
library(sde)
sde.sim(X0=X0, N=N, M=M, t0=t0, T=T, theta=c(theta1, theta2, theta3),
model="CIR") -> X

dt=(T-t0)/N

X.mean = rowMeans(X)
X.sd    = apply(X,1,sd)

plot(as.vector(time(X)),X.mean,type="l",xlab="time",ylab="value")
lines(as.vector(time(X)),X.mean + (1.96*X.sd)/sqrt(M))
lines(as.vector(time(X)),X.mean - (1.96*X.sd)/sqrt(M))
abline(h=2, col = "red")
```

Simulation of the log matrix of the doubly stochastic survival probability with a random intensity.

```
X0=10
N=100
t0=0
T=12
M=1000
theta1=6
theta2=3
theta3=0.5

library(sde)
X <- sde.sim(X0=X0, N=N, M=M, t0=t0, T=T, theta=c(theta1, theta2, theta3),
model="CIR")

dt=(T-t0)/N

default <- function (i,j,X,dt) {
  if (j <= i) {
    return(1)
  }

  if (j == i+1) {
    return (mean(exp(-dt*X[j,])))
  }

  return (mean(exp(-dt*colSums(X[(i+1):j,]))))
}

l = matrix(1,N,N)

for (i in 1:N) {
  for (j in 1:N) {
    l[i,j] = default(i, j, X, dt)
  }
}

image (1:N,1:N,log(l))
```

Figure 2. Simulation of the default model with a smaller κ .

Example of one simulation of CIR process.

```
X0=10
N=100
t0=0
T=12
M=1
theta1=1
theta2=0.5
theta3=0.5
library(sde)
```

```
sde.sim(X0=X0, N=N, M=M, t0=t0, T=T, theta=c(theta1, theta2, theta3),
model="CIR") -> X
plot (X)
abline(h=2, col = "red")
```

The survival probability with deterministic intensity λ_t

```
X0=10
N=100
t0=0
T=12
M=1000
theta1=1
theta2=0.5
theta3=0.5
library(sde)
sde.sim(X0=X0, N=N, M=M, t0=t0, T=T, theta=c(theta1, theta2, theta3),
model="CIR") -> X

dt=(T-t0)/N

X.mean = rowMeans(X)
X.sd    = apply(X,1,sd)

plot(as.vector(time(X)),X.mean,type="l",xlab="time",ylab="value")
lines(as.vector(time(X)),X.mean + (1.96*X.sd)/sqrt(M))
lines(as.vector(time(X)),X.mean - (1.96*X.sd)/sqrt(M))
```

Simulation of the log matrix of the doubly stochastic survival probability with a random intensity.

```
X0=10
N=100
t0=0
T=12
M=1000
theta1=1
theta2=0.5
theta3=0.5

library(sde)
X <- sde.sim(X0=X0, N=N, M=M, t0=t0, T=T, c(theta1, theta2, theta3),
model="CIR")

dt=(T-t0)/N

default <- function (i,j,X,dt) {
  if (j <= i) {
    return(1)
  }

  if (j == i+1) {
    return (mean(exp(-dt*X[j,])))
  }

  return (mean(exp(-dt*colSums(X[(i+1):j,]))))
}
```

```

l = matrix(1,N,N)

for (i in 1:N) {
  for (j in 1:N) {
    l[i,j] = default(i, j, X, dt)
  }
}

image (1:N,1:N,log(l))

```

Figure 3. Simulation of the default model with an augmented κ .

Example of one simulation of CIR process.

```

X0=10
N=100
t0=0
T=12
M=1
theta1=20
theta2=10
theta3=0.5
library(sde)
sde.sim(X0=X0, N=N, M=M, t0=t0, T=T, theta=c(theta1, theta2, theta3),
model="CIR") -> X
plot (X)

```

The survival probability with deterministic intensity λ_t

```

X0=10
N=100
t0=0
T=12
M=1000
theta1=20
theta2=10
theta3=0.5
library(sde)
sde.sim(X0=X0, N=N, M=M, t0=t0, T=T, theta=c(theta1, theta2, theta3),
model="CIR") -> X

dt=(T-t0)/N

X.mean = rowMeans(X)
X.sd    = apply(X,1,sd)

plot(as.vector(time(X)),X.mean,type="l",xlab="time",ylab="value")
lines(as.vector(time(X)),X.mean + (1.96*X.sd)/sqrt(M))
lines(as.vector(time(X)),X.mean - (1.96*X.sd)/sqrt(M))

```

Simulation of the log matrix of the doubly stochastic survival probability with a random intensity.

```

X0=10
N=100
t0=0
T=12
M=1000
theta1=20
theta2=10
theta3=0.5

library(sde)
X <- sde.sim(X0=X0, N=N, M=M, t0=t0, T=T, theta=c(theta1, theta2, theta3),
model="CIR")

dt=(T-t0)/N

default <- function (i,j,X,dt) {
  if (j <= i) {
    return(1)
  }

  if (j == i+1) {
    return (mean(exp(-dt*X[j,])))
  }

  return (mean(exp(-dt*colSums(X[(i+1):j,]))))
}

l = matrix(1,N,N)

for (i in 1:N) {
  for (j in 1:N) {
    l[i,j] = default(i, j, X, dt)
  }
}

image (1:N,1:N,log(l))

```

FIGURE 4 : SIMULATION OF THE DEFAULT MODEL WITH A REDUCED θ .

Example of one simulation of CIR process.

```

X0=10
N=100
t0=0
T=12
M=1
theta1=0.05
theta2=0.5
theta3=0.5
library(sde)

```



```
sde.sim(X0=X0, N=N, M=M, t0=t0, T=T, theta=c(theta1, theta2, theta3),
model="CIR") -> X
plot (X)
```

The survival probability with deterministic intensity λ_t

```
X0=10
N=100
t0=0
T=12
M=1000
theta1=0.05
theta2=0.5
theta3=0.5
library(sde)
sde.sim(X0=X0, N=N, M=M, t0=t0, T=T, theta=c(theta1, theta2, theta3),
model="CIR") -> X
```

```
dt=(T-t0)/N
```

```
X.mean = rowMeans(X)
X.sd    = apply(X,1,sd)
```

```
plot(as.vector(time(X)),X.mean,type="l",xlab="time",ylab="value")
lines(as.vector(time(X)),X.mean + (1.96*X.sd)/sqrt(M))
lines(as.vector(time(X)),X.mean - (1.96*X.sd)/sqrt(M))
```

Simulation of the log matrix of the doubly stochastic survival probability with a random intensity.

```
X0=10
N=100
t0=0
T=12
M=1000
theta1=0.05
theta2=0.5
theta3=0.5

library(sde)
X <- sde.sim(X0=X0, N=N, M=M, t0=t0, T=T, theta=c(theta1, theta2, theta3),
model="CIR")
```

```
dt=(T-t0)/N
```

```
default <- function (i,j,X,dt) {
  if (j <= i) {
    return(1)
  }

  if (j == i+1) {
    return (mean(exp(-dt*X[j,])))
  }

  return (mean(exp(-dt*colSums(X[(i+1):j,]))))
}
```

```

l = matrix(1,N,N)

for (i in 1:N) {
  for (j in 1:N) {
    l[i,j] = default(i, j, X, dt)
  }
}

image (1:N,1:N,log(l))

```

FIGURE 5 : SIMULATION OF THE DEFAULT MODEL WITH AN INCREASE OF θ .

Example of one simulation of CIR process.

```

X0=10
N=100
t0=0
T=12
M=1
theta1=4.5
theta2=0.5
theta3=0.5
library(sde)
sde.sim(X0=X0, N=N, M=M, t0=t0, T=T, theta=c(theta1, theta2, theta3),
model="CIR") -> X
plot (X)

```

The survival probability with deterministic intensity λ_t

```

X0=10
N=100
t0=0
T=12
M=1000
theta1=4.5
theta2=0.5
theta3=0.5
library(sde)
sde.sim(X0=X0, N=N, M=M, t0=t0, T=T, theta=c(theta1, theta2, theta3),
model="CIR") -> X

dt=(T-t0)/N

X.mean = rowMeans(X)
X.sd    = apply(X,1,sd)

plot(as.vector(time(X)),X.mean,type="l",xlab="time",ylab="value")
lines(as.vector(time(X)),X.mean + (1.96*X.sd)/sqrt(M))
lines(as.vector(time(X)),X.mean - (1.96*X.sd)/sqrt(M))

```

Simulation of the log matrix of the doubly stochastic survival probability with a random intensity.

```
X0=10
N=100
t0=0
T=12
M=1000
theta1=4.5
theta2=0.5
theta3=0.5

library(sde)
X <- sde.sim(X0=X0, N=N, M=M, t0=t0, T=T, theta=c(theta1, theta2, theta3),
model="CIR")

dt=(T-t0)/N

default <- function (i,j,X,dt) {
  if (j <= i) {
    return(1)
  }

  if (j == i+1) {
    return (mean(exp(-dt*X[j,])))
  }

  return (mean(exp(-dt*colSums(X[(i+1):j,]))))
}

l = matrix(1,N,N)

for (i in 1:N) {
  for (j in 1:N) {
    l[i,j] = default(i, j, X, dt)
  }
}

image (1:N,1:N,log(l))
```

FIGURE 6. SIMULATION OF THE DEFAULT MODEL WITH DUFFIE'S (1999) PARAMETERS ESTIMATES.

Example of one simulation of CIR process.

```
X0=10
N=100
t0=0
T=12
M=1
theta1=0.559
```

```

theta2=0.238
theta3=0.074
library(sde)
sde.sim(X0=X0, N=N, M=M, t0=t0, T=T, theta=c(theta1, theta2, theta3),
model="CIR") -> X
plot (X)

```

The survival probability with deterministic intensity λ_t

```

X0=10
N=100
t0=0
T=12
M=1000
theta1=0.559
theta2=0.238
theta3=0.074
library(sde)
sde.sim(X0=X0, N=N, M=M, t0=t0, T=T, theta=c(theta1, theta2, theta3),
model="CIR") -> X

dt=(T-t0)/N

X.mean = rowMeans(X)
X.sd    = apply(X,1,sd)

plot(as.vector(time(X)),X.mean,type="l",xlab="time",ylab="value")
lines(as.vector(time(X)),X.mean + (1.96*X.sd)/sqrt(M))
lines(as.vector(time(X)),X.mean - (1.96*X.sd)/sqrt(M))

```

Simulation of the log matrix of the doubly stochastic survival probability with a random intensity.

```

X0=10
N=100
t0=0
T=12
M=1000
theta1=0.559
theta2=0.238
theta3=0.074

library(sde)
X <- sde.sim(X0=X0, N=N, M=M, t0=t0, T=T, theta=c(theta1, theta2, theta3),
model="CIR")

dt=(T-t0)/N

default <- function (i,j,X,dt) {
  if (j <= i) {
    return(1)
  }

  if (j == i+1) {
    return (mean(exp(-dt*X[j,])))
  }
}

```

```

    return (mean(exp(-dt*colSums(X[(i+1):j,])))
  }

l = matrix(1,N,N)

for (i in 1:N) {
  for (j in 1:N) {
    l[i,j] = default(i, j, X, dt)
  }
}

image (1:N,1:N,log(l))

```

FIGURE 7. SIMULATION OF THE DEFAULT MODEL WITH BRIGO AND AFONSI'S (2005) PARAMETERS ESTIMATES.

Example of one simulation of CIR process.

```

X0=10
N=100
t0=0
T=12
M=1
theta1=0.00125
theta2=0.25
theta3=0.1
library(sde)
sde.sim(X0=X0, N=N, M=M, t0=t0, T=T, theta=c(theta1, theta2, theta3),
model="CIR") -> X
plot (X)

```

The survival probability with deterministic intensity λ_t

```

X0=10
N=100
t0=0
T=12
M=1000
theta1=0.00125
theta2=0.25
theta3=0.1
library(sde)
sde.sim(X0=X0, N=N, M=M, t0=t0, T=T, theta=c(theta1, theta2, theta3),
model="CIR") -> X

dt=(T-t0)/N

X.mean = rowMeans(X)
X.sd    = apply(X,1,sd)

plot(as.vector(time(X)),X.mean,type="l",xlab="time",ylab="value")
lines(as.vector(time(X)),X.mean + (1.96*X.sd)/sqrt(M))

```

```
lines(as.vector(time(X)),X.mean - (1.96*X.sd)/sqrt(M))
```

Simulation of the log matrix of the doubly stochastic survival probability with a random intensity.

```
X0=10
N=100
t0=0
T=12
M=1000
theta1=0.00125
theta2=0.25
theta3=0.1

library(sde)
X <- sde.sim(X0=X0, N=N, M=M, t0=t0, T=T, theta=c(theta1, theta2, theta3),
model="CIR")

dt=(T-t0)/N

default <- function (i,j,X,dt) {
  if (j <= i) {
    return(1)
  }

  if (j == i+1) {
    return (mean(exp(-dt*X[j,])))
  }

  return (mean(exp(-dt*colSums(X[(i+1):j,]))))
}

l = matrix(1,N,N)

for (i in 1:N) {
  for (j in 1:N) {
    l[i,j] = default(i, j, X, dt)
  }
}

image (1:N,1:N,log(l))
```

FIGURE 8. SIMULATION OF THE LOGARITHM OF THE DOUBLY STOCHASTIC DEFAULT WITH LOW CIR PARAMETERS.

```
X0=10
N=100
t0=0
T=12
M=1000
theta1=0.0000001
theta2=0.0001
theta3=1

library(sde)
```

```

X <- sde.sim(X0=X0, N=N, M=M, t0=t0, T=T, theta=c(theta1, theta2, theta3),
model="CIR")

dt=(T-t0)/N

default <- function (i,j,X,dt) {
  if (j <= i) {
    return(1)
  }

  if (j == i+1) {
    return (mean(exp(-dt*X[j,])))
  }

  return (mean(exp(-dt*colSums(X[(i+1):j,]))))
}

l = matrix(1,N,N)

for (i in 1:N) {
  for (j in 1:N) {
    l[i,j] = default(i, j, X, dt)
  }
}

image (1:N,1:N,log(l))

```

FIGURE 9. SIMULATION OF THE MATRIX OF THE DOUBLY STOCHASTIC DEFAULT WITH LOW CIR PARAMETERS.

```

N=100
t0=0
T=12
M=1000
theta1=0.0000001
theta2=0.0001
theta3=1

library(sde)
X <- sde.sim(X0=X0, N=N, M=M, t0=t0, T=T, theta=c(theta1, theta2, theta3),
model="CIR")

dt=(T-t0)/N

default <- function (i,j,X,dt) {
  if (j <= i) {
    return(1)
  }

  if (j == i+1) {
    return (mean(exp(-dt*X[j,])))
  }

  return (mean(exp(-dt*colSums(X[(i+1):j,]))))
}

```

```

l = matrix(1,N,N)

for (i in 1:N) {
  for (j in 1:N) {
    l[i,j] = default(i, j, X, dt)
  }
}

image (1:N,1:N,l)

```

FIGURE 10. SIMULATION WITH LOWER VOLATILITY OF THE LEVEL CURVE OF DOUBLY STOCHASTIC PROBABILITY.

```

X0=10
N=100
t0=0
T=12
M=1000
theta1=0.0000001
theta2=0.0001
theta3=0.1

library(sde)
X <- sde.sim(X0=X0, N=N, M=M, t0=t0, T=T, theta=c(theta1, theta2, theta3),
model="CIR")

dt=(T-t0)/N

default <- function (i,j,X,dt) {
  if (j <= i) {
    return(1)
  }

  if (j == i+1) {
    return (mean(exp(-dt*X[j,])))
  }

  return (mean(exp(-dt*colSums(X[(i+1):j,]))))
}

l = matrix(1,N,N)

for (i in 1:N) {
  for (j in 1:N) {
    l[i,j] = default(i, j, X, dt)
  }
}

image (1:N,1:N,l)

```


FIGURE 11. SIMULATION OF ONE PATH OF THE CIR PROCESS FOR RISK-FREE INTEREST RATE.

```
X0a=10
Na=100
t0a=0
Ta=12
Ma=1000
thetaa=c(0.514878, 0.082, 0.67)
library(sde)
Y <- sde.sim(X0=X0a, N=Na, M=Ma, t0=t0a, T=Ta, theta=thetaa, model="CIR")
Plot(Y, ylab=r)
abline(h=6.279, col="red")
```

FIGURE 12. SIMULATION OF THE LOGARITHM DISCOUNTED EXPECTED CASH FLOW.

```
X0=10
N=100
t0=0
T=12
M=1000
theta=c(0.559, 0.238, 0.074)

X0a=10
Na=100
t0a=0
Ta=12
Ma=1000
thetaa=c(0.514878, 0.082, 0.67)
library(sde)
X <- sde.sim(X0=X0, N=N, M=M, t0=t0, T=T, theta=theta, model="CIR")

Y <- sde.sim(X0=X0a, N=Na, M=Ma, t0=t0a, T=Ta, theta=thetaa, model="CIR")

W=X+Y

dt=(T-t0)/N
W.mean = rowMeans(W)
W.sd = apply(W,1,sd)
plot(as.vector(time(W)),W.mean,type="l",xlab="time",ylab="value")
lines(as.vector(time(W)),W.mean + (1.96*W.sd)/sqrt(M))
lines(as.vector(time(W)),W.mean - (1.96*W.sd)/sqrt(M))
```

FIGURE 13. SIMULATION OF THE LOGARITHM PRICE OF A ZERO COUPON BOND BEFORE DEFAULT.

```
X0=10
N=100
t0=0
```

```

T=12
M=1000
theta=c(0.559, 0.238, 0.074)

X0a=10
Na=100
t0a=0
Ta=12
Ma=1000
thetaa=c(0.514878, 0.0082, 0.67)
library(sde)
X <- sde.sim(X0=X0, N=N, M=M, t0=t0, T=T, theta=theta, model="CIR")

Y <- sde.sim(X0=X0a, N=Na, M=Ma, t0=t0a, T=Ta, theta=thetaa, model="CIR")

W=X+Y

dt=(T-t0)/N
W.mean = rowMeans(W)
W.sd    = apply(W,1,sd)
plot(as.vector(time(W)),W.mean,type="l",xlab="time",ylab="value")
lines(as.vector(time(W)),W.mean + (1.96*W.sd)/sqrt(M))
lines(as.vector(time(W)),W.mean - (1.96*W.sd)/sqrt(M))

default <- function (i,j,W,dt) {
  if (j <= i) {
    return(1)
  }

  if (j == i+1) {
    return (mean(exp(-dt*W[j,])))
  }

  return (mean(exp(-dt*colSums(W[(i+1):j,]))))
}

l = matrix(1,N,N)

for (i in 1:N) {
  for (j in 1:N) {
    l[i,j] = default(i, j, W, dt)
  }
}

image (1:N,1:N,log(l))

```

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